DISCREPANCY SKEW PRODUCTS AND AFFINE RANDOM WALKS

JON. AARONSON , MICHAEL BROMBERG AND HITOSHI NAKADA

ABSTRACT. We prove bounded rational ergodicity for some discrepancy skew products whose rotation number has bad rational approximation. This is done by considering the asymptotics of associated affine random walks.

§1 INTRODUCTION

Discrepancy skew products. Let $\mathbb{T} := \mathbb{R}/\mathbb{Z} \cong [0,1)$ denote the additive circle.

Consider the function $\varphi : \mathbb{T} \to \mathbb{Z}$ defined by

$$\varphi := 2 \cdot 1_{[0,\frac{1}{2})} - 1$$

and the *skew products* $T_\alpha : \mathbb{T} \times \mathbb{Z} \to \mathbb{T} \times \mathbb{Z}$ defined for $\alpha \notin \mathbb{Q}$ by

$$T_\alpha(x,y) = (x+\alpha, y+\varphi(x)).$$

These are measure preserving transformations of the σ-finite measure space

$$(X, \mathcal{B}, m) = (\mathbb{T} \times \mathbb{Z}, \mathcal{B}(\mathbb{T} \times \mathbb{Z}), \mathtt{Leb} \times \#).$$

We have that $T_\alpha^n(x,y) = (x+n\alpha, y+\varphi_n(x))$ where

$$\varphi_n(x) = \sum_{k=0}^{n-1} \varphi(x+k\alpha).$$

This is related to the discrepancy of the well-distribution of $(\{n\alpha\})_{n\ge1}$ over $[0,\frac{1}{2})$ (see [16]) and accordingly we call the function $(n,x) \mapsto \varphi_n(x)$ the *discrepancy cocycle* and T_α the *discrepancy skew product*

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(aka the "cylinder flow arising from irregularity of distribution" in [16] and "deterministic random walk" as in [3]).

Ergodicity of the discrepancy skew product T_α was established for $\alpha = \frac{\sqrt{5}-1}{4}$ in [16] and then $\forall\ \alpha \in \mathbb{R} \setminus \mathbb{Q}$ in [8].

Results.

We call $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *badly approximable* if

$$\inf \{q^2|\alpha - \tfrac{p}{q}| :\ q \in \mathbb{N},\ p \in \mathbb{Z}\} > 0$$

and denote

$$\mathtt{BAD} := \{\alpha \in \mathbb{R} \setminus \mathbb{Q} :\ \alpha \ \text{ badly approximable}\}.$$

Our main result is that if $\alpha \in \mathtt{BAD}$, then T_α is *boundedly rationally ergodic* in the sense of [1]. In particular, defining $\Psi_n = \Psi_n^{(\alpha)} : \mathbb{T} \to \mathbb{N}$ by

$$\begin{aligned}\Psi_n(x) &= S_n(1_{\mathbb{T}\times\{0\}})(x,0)\\ &:= \sum_{k=0}^{n-1} 1_{\mathbb{T}\times\{0\}} \circ T_\alpha^k(x,0)\\ &= \#\{0 \le k \le n-1 :\ \varphi_k(x) = 0\}\end{aligned}$$

we prove that $\exists\ M > 1$ so that

$$(🚂)\qquad \|\Psi_n^{(\alpha)}\|_{L^\infty(\mathbb{T})} \le M \int_{\mathbb{T}} \Psi_n^{(\alpha)}(t)dt \ \text{ and } \ \int_{\mathbb{T}} \Psi_n^{(\alpha)}(t)dt = M^{\pm 1} \frac{n}{\sqrt{\log n}}.$$

Here and throughout, for $a, b > 0,\ M > 1$,

$$a = M^{\pm 1} b \text{ means } \ \frac{1}{M} \le \frac{a}{b} \le M.$$

We'll also consider more general subsequence versions of (🚂):

$$(☸)\qquad \|\Psi_{\ell_n}^{(\alpha)}\|_{L^\infty(\mathbb{T})} \le M \int_{\mathbb{T}} \Psi_{\ell_n}^{(\alpha)}(t)dt$$

which implies (as in [1]) that there is a dense hereditary ring $R(T_\alpha)$ of sets of finite measure so that

$$\sum_{j=0}^{\ell_n - 1} m(A \cap T^{-j}B) \ \sim\ m(A)m(B)a_{\ell_n} \text{ as } n \to \infty\ \forall\ A,\ B \in R(T)$$

where $a_{\ell_n} := \int_{\mathbb{T}} \Psi_{\ell_n}^{(\alpha)}(t)dt$.

In case $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ quadratic, (🚂) was established in [3] and refined in [4].

The discrepancy skew products occur as `good sections` (in the sense of [2]) for directional translation flows of the infinite staircase translation surface as in [13] (see [4]). Our result also holds for the corresponding directional, translation flow (with α badly approximable) as can be seen using lemma 2.1 in [2].

Proof perspective.

The proof of (🚗) relies on a weak, rough local limit theorem for an associated affine random walk arising from sequence of renormalizations related to the orbit of 2α under a modified continued fraction transformation (see below).

The quadratic case corresponds to a compact (eventually periodic) renormalization sequence, and the `BAD` case corresponds to a precompact sequence.

`RAT`s and `ARW`s. A *random affine transformation* (`RAT`) on $\mathbb{R}^d$ is a random variable $F = (a(F), b(F))$ taking values in $M_{d\times d}(\mathbb{R}) \times \mathbb{R}^d$ (associated with the transformation $x \mapsto ax + b$).

We call the `RAT` F *discrete* if $F \in M_{d\times d}(\mathbb{Z}) \times \mathbb{Z}^d$ a.s.

An *affine random walk* (`ARW`) is a $\mathbb{R}^d$-valued stochastic process $(X^{(n)})_{n\ge 0}$ defined by

$$X^{(0)} := 0 \text{ and } X^{(n+1)} := F_{n+1}(X^{(n)}) = a(F_n)X^{(n)} + b(F_n)$$

where the $(F_n : n \ge 1)$ is a sequence of independent `RAT`s referred to as the `RAT` *sequence*. In this paper, we only have need of a special kind of `RAT` called `flip type` (defined in §4). For other works on `ARW`s (not of flip type), see [10], [14], [17], [11], [9] and references therein.

The rational ergodicity of the discrepancy skew product is governed by the asymptotic behaviour of the temporal statistics of the discrepancy cocycle values $(\varphi_n(0) : n \ge 1)$ (Visit Lemma 2.4 below). These temporal statistics are modelled by certain `ARW`s (Construction Lemma 4.1 below) and (🚗) follows from a weak, rough, local limit theorem for the coordinates of these (Theorem 6.2 below).

§2 Visit distributions of the discrepancy cocycle

In this section, as in [3], we show that (🚗) (as on page 2) follows from certain asymptotic properties of "`visit distributions`" (to be defined below).

We first note that we may assume without loss of generality that $0 < \alpha < \frac{1}{2}$.

This is because $-T_\alpha(-x,-n) = T_{1-\alpha}(x,n)$ and $-\varphi_k^{(\alpha)}(x) = \varphi_k^{(1-\alpha)}(-x)$ whence

$$\Psi_n^{(\alpha)}(x) = \Psi_n^{(1-\alpha)}(-x).$$

Thus (🚗) (as on page 2) for α and $1-\alpha$ are equivalent and we only consider the case $0 < \alpha < \frac{1}{2}$.

Calculation of the jump function orbit.

We recall from [3] the substitution algorithm to calculate the jump function orbit $(\varphi(\{n\alpha\}): \ n \geq 0)$.

We have,

$$\varphi(n\alpha) = (-1)^{\sum_{j=1}^n \psi_j}$$

where

$$\psi_n = \psi_n^{(2\alpha)} := 1_{[1-2\alpha,1)}(2(n-1)\alpha).$$

The *modified continued fraction expansion* of $\beta \in (0,1)$ is

$$\beta = [n_1, n_2, \dots] =: 1/n_1 - 1/n_2 - 1/n_3 - \dots$$

with each $n_k \in \mathbb{N}_2 := \{a \in \mathbb{N} : \ a \geq 2\}$. Here (see [15]) $\beta \in \mathbb{Q}$ iff $n_k \to 2$ as $n \to \infty$.

Theorem 2.1 ([3]) *For $\beta = [n_1, n_2, \dots]$, let $b_0(0) = 0, \ b_0(1) = 1$ and*

$$b_{k+1}(0) = b_k(0)^{\odot(n_{k+1}-1)} \odot b_k(1) \quad \text{and} \quad b_{k+1}(1) = b_k(0))^{\odot(n_{k+1}-2)} \odot b_k(1),$$

then

$$(\psi_1^{(\beta)}, \dots, \psi_{\ell_k(0)}^{(\beta)}) = b_k(0) \quad (k \geq 1).$$

Here $\odot$ denotes concatenation, and $\ell_k(i) = |b_k(i)|$ denotes the length of the block $b_k(i)$ $(i = 0,1)$.

For $\beta = 2\alpha = [n_1, n_2, \dots]$, set for $i = 0,1$, $B_0(i) := [(-1)^i]$ and for $k \geq 0$, $i = 0,1$ and $(n_{k+1}, i) \neq (2,1)$:

$$B_{k+1}(i) = \bigodot_{j=1}^{n_{k+1}-1-i} (-1)^{(j-1)\epsilon_k(0)} B_k(0) \odot (-1)^{\epsilon_k(0)(n_{k+1}-1-i)} B_k(1);$$

where

$$\epsilon_k(i) := \sum_{j=1}^{\ell_k(i)} (b_k(i))_j \mod 2$$

and $B_{k+1}(1) = B_k(1)$ in case $n_{k+1} = 2$.

Theorem 2.2 ([3])

$$B_k(0) = (\varphi(\{j\alpha\}))_{j=0}^{\ell_k(0)-1} \quad (i = 0,1).$$

Visit sets.

The *visit set to* $\nu \in \mathbb{Z}$ is

$$K_\nu := \{n \geq 1 : \ \varphi_n(0) = \nu\}$$

and the *visit distributions* are the measures $U_k^{(i)}$ on $\mathbb{Z}$ defined by

$$U_k^{(i)}(\nu) := \#(K_\nu \cap [1, \ell_k(i)]) \quad (k \geq 1, \ i = 0, 1).$$

Lemma 2.3

$$\int_0^1 \Psi_{\ell_k(0)}(x)dx \ \geq \ \frac{1}{4\ell_k(0)} \sum_{\nu \in \mathbb{Z}} [U_k^{(1)}(\nu)]^2; \tag{4.1}$$

$$\int_0^1 \Psi_{\ell_k(1)}(x)^N dx \ \leq \ \frac{2^N}{\ell_k(1)} \sum_{\nu \in \mathbb{Z}} [U_k^{(0)}(\nu)]^{N+1} \quad \forall \ N \geq 1. \tag{4.2}$$

Statement (4.1) is essentially Lemma 4.1 in [3] and proved in the same manner. Statement (4.2) is an upgrade of lemma 4.2 in [3].

Proof of (4.2) As in the proofs of lemmas 4.1 and 4.2 in [3],

$$\ell_{k+r}(0) \int_0^1 \Psi_{\ell_k(1)}(x)^N dx \underset{r \to \infty}{\sim} \sum_{\nu \in \mathbb{Z}} \sum_{j \in [1, \ell_{k+r}(0)] \cap K_\nu} \#\left(K_\nu \cap [j+1, j+\ell_k(1)]\right)^N$$

and for $r \geq 1, \ \exists \ J = J_{r,k} \geq 1, \ 1 = m_1 < \cdots < m_J$ and $\epsilon_1, \ldots \epsilon_{J-1} = \pm 1, \ i_1, \ldots, i_{J-1} = 0, 1$ so that

$$m_{j+1} - m_j = \ell_k(i_j) \ \forall \ j, \quad [1, \ell_{k+r}(0)] = \bigcupdot_{j=1}^{J-1} [m_j, m_{j+1}),$$

$$(\varphi(m_j\alpha), \varphi(m_j+1)\alpha), \ldots, \varphi((m_{j+1}-1)\alpha)) = \epsilon_j B_k(i_j).$$

For fixed $\nu \in \mathbb{Z}$,

$$K_\nu \cap [m_j, m_{j+1}) = m_j + K_{\epsilon_j(\nu - \varphi_{m_j}(0))} \cap [1, \ell_k(i_j)].$$

Note that

$$(\varphi(m_j\alpha), \varphi((m_j+1)\alpha), \ldots, \varphi((m_{j+1}+\ell_k(1)-1)\alpha)) = (\epsilon_j B_k(i_j), \Delta_j B_k(1))$$

for some $\Delta_j = \pm 1$.

We have as before, for fixed $\nu \in \mathbb{Z}$,

$$\begin{aligned}\sum_{i\in[1,\ell_{k+r}(0)]\cap K_\nu} \#\left(K_\nu \cap [i+1, i+\ell_k(1)]\right)^N &= \sum_{j=1}^{J-1} \sum_{i\in[m_j,m_{j+1})\cap K_\nu} \#\left(K_\nu \cap [i+1, i+\ell_k(1)]\right)^N \\ &\leq \sum_{j=1}^{J-1} \sum_{i\in[m_j,m_{j+1})\cap K_\nu} \#\left(K_\nu \cap [i+1, m_{j+1}+\ell_k(1))]\right)^N.\end{aligned}$$

Fix j. For fixed $i \in [m_j, m_{j+1})$,

$$\begin{aligned}\#\left(K_\nu \cap [i+1, m_{j+1}+\ell_k(1))]\right) &= \#\left(K_\nu \cap [i+1, m_{j+1}]\right) + \#\left(K_\nu \cap [m_{j+1}+1, m_{j+1}+\ell_k(1))]\right) \\ &= \#\left(K_\nu \cap [i+1, m_{j+1}]\right) + U_k^{(1)}(\Delta_j(\nu - \varphi_{m_{j+1}(0)})).\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{i\in[m_j,m_{j+1})\cap K_\nu} \#\left(K_\nu \cap [i+1, m_{j+1}+\ell_k(1))]\right)^N = \\ &= \sum_{i\in[m_j,m_{j+1})\cap K_\nu} \left(\#\left(K_\nu \cap [i+1, m_{j+1}]\right) + U_k^{(1)}(\Delta_j(\nu - \varphi_{m_{j+1}(0)}))\right)^N \\ &= \sum_{r=0}^{N} \binom{N}{r} \left(\sum_{i\in[m_j,m_{j+1})\cap K_\nu} \#\left(K_\nu \cap [i+1, m_{j+1}]\right)^r \right) U_k^{(1)}(\Delta_j(\nu - \varphi_{m_{j+1}(0)}))^{N-r} \\ &\leq \sum_{r=0}^{N} \binom{N}{r} U_k^{(i_j)}(\epsilon_j(\nu - \varphi_{m_j(0)}))^{r+1} U_k^{(1)}(\Delta_j(\nu - \varphi_{m_{j+1}(0)}))^{N-r} \\ &\leq \sum_{r=0}^{N} \binom{N}{r} U_k^{(0)}(\epsilon_j(\nu - \varphi_{m_j(0)}))^{r+1} U_k^{(0)}(\Delta_j(\nu - \varphi_{m_{j+1}(0)}))^{N-r}.\end{aligned}$$

Using this and Hölder's inequality,

$$\begin{aligned}&\sum_{\nu\in\mathbb{Z}} \sum_{j\in[1,\ell_{k+r}(0)]\cap K_\nu} \#\left(K_\nu \cap [j+1, j+\ell_k(1)]\right)^N \leq \\ &\leq \sum_{j=1}^{J} \sum_{r=0}^{N} \binom{N}{r} \sum_{\nu\in\mathbb{Z}} U_k^{(0)}(\epsilon_j(\nu - \varphi_{m_j(0)}))^{r+1} U_k^{(0)}(\Delta_j(\nu - \varphi_{m_{j+1}(0)}))^{N-r} \\ &\leq \sum_{j=1}^{J} \sum_{r=0}^{N} \binom{N}{r} \left(\sum_{\nu\in\mathbb{Z}} U_k^{(0)}(\epsilon_j(\nu - \varphi_{m_j(0)}))^{N+1} \right)^{\frac{r+1}{N+1}} \left(\sum_{\nu\in\mathbb{Z}} U_k^{(0)}(\Delta_j(\nu - \varphi_{m_{j+1}(0)}))^{N+1} \right)^{\frac{N-r}{N+1}} \\ &\leq J \sum_{r=0}^{N} \binom{N}{r} \left(\sum_{\nu\in\mathbb{Z}} U_k^{(0)}(\nu)^{N+1} \right)^{\frac{r+1}{N+1}} \left(\sum_{\nu\in\mathbb{Z}} U_k^{(0)}(\nu)^{N+1} \right)^{\frac{N-r}{N+1}} \\ &= 2^N J \sum_{\nu\in\mathbb{Z}} U_k^{(0)}(\nu)^{N+1}\end{aligned}$$

whence

$$\int_0^1 \Psi_{\ell_k(1)}(x)^N dx \xleftarrow[r\to\infty]{} \frac{1}{\ell_{k+r}(0)} \sum_{\nu\in\mathbb{Z}} \sum_{j\in[1,\ell_{k+r}(0)]\cap K_\nu} \#\left(K_\nu \cap [j+1, j+\ell_k(1)]\right)^N$$
$$\le \frac{2^N J}{\ell_{k+r}(0)} \sum_{\nu\in\mathbb{Z}} U_k^{(0)}(\nu)^{N+1}$$
$$\le \frac{2^N}{\ell_k(1)} \sum_{\nu\in\mathbb{Z}} U_k^{(0)}(\nu)^{N+1}. \quad ☑(4.2)$$

Visit lemma

This is a Fourier series (or generating function) consequence of Lemma 2.3.

Let

$$\widehat{U}_k^{(i)}(Z) := \sum_{\nu\in\mathbb{Z}} U_k^{(i)}(\nu) Z^\nu \qquad (i = 0, 1,\ |Z| = 1).$$

Visit lemma 2.4

(4.1') $$\int_0^1 \Psi_{\ell_k(0)}(x)dx \ \ge\ \frac{1}{4\ell_k(0)} \int_{\mathbb{S}^1} |\widehat{U}_k^{(1)}(Z)|^2 dZ$$

(4.2') $$\|\Psi_{\ell_k(1)}\|_\infty \le 2 \int_{\mathbb{S}^1} |\widehat{U}_k^{(1)}(Z)| dZ.$$

Where here and throughout,

$$\int_{\mathbb{S}^1} f(Z)dZ := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})d\theta.$$

Proof For fixed $k \ge 1$,

The statement (4.1') follows from (4.1) via the Riesz-Fischer theorem and the statement (4.2'), follows from (4.2) using the Hausdorff-Young theorem as in the proof of theorem 6.1 of [3]. ☑

§3 Visit distribution transitions (as in [3])

Set $s_k(i) := \sum_{j=0}^{\ell_k(i)-1} \varphi(\{j\alpha\})$ and define the *orbit blocks*

$$\Sigma_k(i) := (\varphi_1(0), \varphi_2(0), \ldots, \varphi_{\ell_k^{(i)}}(0)).$$

Our goal here is to obtain the transitions of the visit distribution generating functions.

Transitions in terms of blocks

From theorem 2.2 above we see that for $k \ge 1$, $i = 0, 1$, where $(n_{k+1}, i) \ne (2, 1)$:

- in case $\epsilon_k(0) = 0$,

$$\Sigma_{k+1}(i) = \bigodot_{j=1}^{n_{k+1}-1-i} (\Sigma_k(0) + (j-1)s_k(0)\mathbb{1}) \odot (\Sigma_k(1) + (n_{k+1} - i - 1)s_k(0)\mathbb{1});$$

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z}$,

$$\Sigma_{k+1}(i) = [\Sigma_k(0), s_k(0)\mathbb{1} - \Sigma_k(0)]^{\odot \frac{n_{k+1}-1-i}{2}} \odot \Sigma_k(1)$$

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z} + 1$,

$$\Sigma_{k+1}(i) = [\Sigma_k(0), s_k(0)\mathbb{1} - \Sigma_k(0)]^{\odot \frac{n_{k+1}-2-i}{2}} \odot \Sigma_k(0) \odot (s_k(0)\mathbb{1} - \Sigma_k(1)).$$

- and $\Sigma_{k+1}(1) = \Sigma_k(1)$ in case $n_{k+1} = 2$.

Transitions in terms of visit distributions

For $k \geq 1$, $i = 0, 1$ and $\nu \in \mathbb{Z}$, where $(n_{k+1}, i) \neq (2, 1)$:

- in case $\epsilon_k(0) = 0$,

$$U_{k+1}^{(i)}(\nu) = \sum_{j=1}^{n_{k+1}-1-i} U_k^{(0)}(\nu - (j-1)s_k(0)) + U_k^{(1)}(\nu - (n_{k+1} - i - 1)s_k(0));$$

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z}$,

$$U_{k+1}^{(i)}(\nu) = \frac{n_{k+1} - 1 - i}{2}(U_k^{(0)}(\nu) + U_k^{(0)}(s_k(0) - \nu)) + U_k^{(1)}(\nu);$$

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z} + 1$,

$$U_{k+1}^{(i)}(\nu) = \frac{n_{k+1} - i}{2} U_k^{(0)}(\nu) + \frac{n_{k+1} - 2 - i}{2} U_k^{(0)}(s_k(0) - \nu)) + U_k^{(1)}(s_k(0) - \nu);$$

- and $U_{k+1}^{(1)}(\nu) = U_k^{(1)}(\nu)$ in case $n_{k+1} = 2$.

Transitions in terms of generating functions

For $k \geq 1$, $i = 0, 1$ and $Z \in \mathbb{S}^1$, where $(n_{k+1}, i) \neq (2, 1)$:

- in case $\epsilon_k(0) = 0$,

$$\widehat{U}_{k+1}^{(i)}(Z) = \sum_{j=1}^{n_{k+1}-1-i} Z^{(j-1)s_k(0)} \widehat{U}_k^{(0)}(Z) + Z^{(n_{k+1}-i-1)s_k(0)} \widehat{U}_k^{(1)}(Z);$$

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z}$,

$$\widehat{U}_{k+1}^{(i)}(Z) = \frac{n_{k+1} - 1 - i}{2}(\widehat{U}_k^{(0)}(Z) + Z^{s_k(0)} \widehat{U}_k^{(0)}(Z^{-1}) + \widehat{U}_k^{(1)}(Z);$$

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z} + 1$,

$$\widehat{U}^{(i)}_{k+1}(\nu) = \frac{n_{k+1} - i}{2}\widehat{U}^{(0)}_k(Z) + \frac{n_{k+1} - 2 - i}{2} Z^{s_k(0)}\widehat{U}^{(0)}_k(Z^{-1}) + Z^{s_k(0)}\widehat{U}^{(1)}_k(Z^{-1});$$

- and $\widehat{U}^{(1)}_{k+1}(Z) = \widehat{U}^{(1)}_k(Z)$ in case $n_{k+1} = 2$.

Simplified visit distributions.
Our next task is to eliminate the dependence of the visit distribution transitions on the positions $s_k(0)$. We'll do this exploiting the parity sequence $(\underline{\epsilon}_k : \ k \geq 1)$.

Parities, block lengths and positions
Let $k \geq 1$. The k^{th}:

$$\textit{parity state vector} \text{ is } \ \underline{\epsilon}_k := \begin{pmatrix} \epsilon_k(0) \\ \epsilon_k(1) \end{pmatrix};$$

$$\textit{block length vector} \text{ is } \ \underline{\ell}_k := \begin{pmatrix} \ell_k(0) \\ \ell_k(1) \end{pmatrix},$$

$$\textit{position vector} \text{ is } \ \underline{s}_k := \begin{pmatrix} s_k(0) \\ s_k(1) \end{pmatrix}$$

where $s_k(i) := \sum_{j=0}^{\ell_k(i)-1} \varphi(\{j\alpha\})$.
Next let

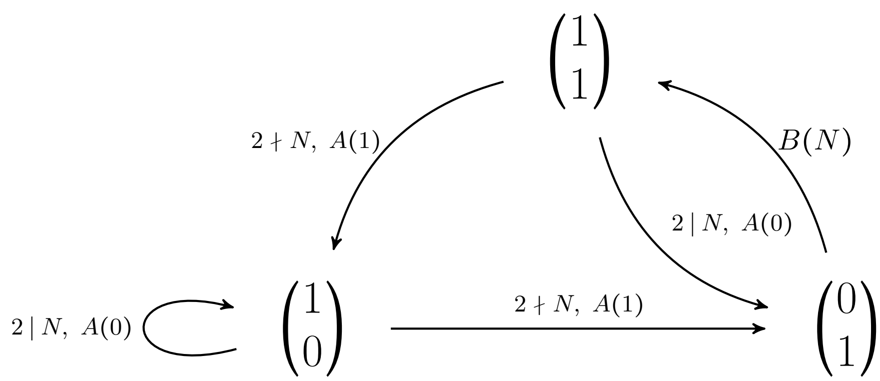
$$A(0) := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad A(1) := \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$B(n) := \begin{pmatrix} n-1 & 1 \\ n-2 & 1 \end{pmatrix} \ \text{ and } A(n) := A(n \mod 1),$$

then

$$\underline{s}_{k+1} = \begin{cases} B(n_{k+1})\underline{s}_k & \epsilon_k(0) = 0; \\ A(n_{k+1})\underline{s}_k & \epsilon_k(0) = 1. \end{cases}$$

The following diagram gives (as in [3]) the parity transitions $\underline{\epsilon}_k \to \underline{\epsilon}_{k+1}$ conditional on $N = n_{k+1}$ together with the corresponding position vector transitions.



Lemma 3.1 ([3]) *For $k \geq 1$,*

$$s_k(0) = 1, \quad s_k(1) = 1 \quad or \quad s_k(0) - s_k(1) = 1$$

according to whether

$$\underline{\epsilon}_k := \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \quad respectively.$$

To this end, define $T_k = T_k(\underline{\epsilon}_k) = T_k(\epsilon_k(0))$ by:

$$T_k\begin{pmatrix}1\\0\end{pmatrix} = T_k\begin{pmatrix}1\\1\end{pmatrix} = -s_k(0), \quad T_k\begin{pmatrix}0\\1\end{pmatrix} = -s_k(1).$$

Evidently

$$(\times) \qquad \underline{\epsilon}_k(0) = 1 \overset{T_k = -s_k(0)}{\Longrightarrow} T_{k+1} - T_k = T_{k+1} + T_k + 2s_k(0)$$

The following is established by straightrforward computation.

Proposition 3.2: Increments of the T_ks

(1) $$T_{k+1}\begin{pmatrix}0\\1\end{pmatrix} - T_k\begin{pmatrix}1\\0\end{pmatrix} = 1,$$

(2) $$T_{k+1}\begin{pmatrix}1\\0\end{pmatrix} - T_k\begin{pmatrix}1\\0\end{pmatrix} = 1,$$

(3) $$T_{k+1}\begin{pmatrix}1\\0\end{pmatrix} - T_k\begin{pmatrix}1\\1\end{pmatrix} = 1$$

(4) $$T_{k+1}\begin{pmatrix}0\\1\end{pmatrix} - T_k\begin{pmatrix}1\\1\end{pmatrix} = 1$$

(5) $$T_{k+1}\begin{pmatrix}1\\1\end{pmatrix} - T_k\begin{pmatrix}0\\1\end{pmatrix} = -(n_{k+1} - 1).$$

Simplified visit distribution transitions.

Given $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ we define the *simplified visit distributions* by

$$V_k^{(i)}(J) := \begin{cases} U_k^{(i)}(\nu) & J = 2\nu + T_k,\ \nu \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases}$$

where $T_k = T_k(\epsilon_k(0))$.

The generating functions are given by

$$\widehat{V}_k^{(i)}(Z) = Z^{T_k}\widehat{U}_k^{(i)}(Z^2).$$

They have the property that

$$\int_{\mathbb{S}^1} |\widehat{V}_k^{(i)}(Z)|^p dZ = \int_{\mathbb{S}^1} |\widehat{U}_k^{(i)}(Z)|^p dZ \ \forall\ p > 0$$

and satisfy simpler recursions as follows:

For $k \geq 1$, $i = 0, 1$ and $Z \in \mathbb{S}^1$, where $(n_{k+1}, i) \neq (2, 1)$:

- in case $\epsilon_k(0) = 0$ we have $s_k(0) = 1$ and $T_{k+1} - T_k = -(n_{k+1} - 1)$ and,

$$\begin{aligned}
\widehat{V}_{k+1}^{(i)}(Z) &= Z^{T_{k+1}}\widehat{U}_{k+1}^{(i)}(Z^2) \\
&= \sum_{j=1}^{n_{k+1}-1-i} Z^{2(j-1)+T_{k+1}}\widehat{U}_k^{(0)}(Z^2) + Z^{2(n_{k+1}-i-1)+T_{k+1}}\widehat{U}_k^{(1)}(Z^2) \\
&= \sum_{j=1}^{n_{k+1}-1-i} Z^{2(j-1)+T_{k+1}-T_k}\widehat{V}_k^{(0)}(Z) + Z^{2(n_{k+1}-i-1)+T_{k+1}-T_k}\widehat{V}_k^{(1)}(Z) \\
&= \sum_{j=1}^{n_{k+1}-1-i} Z^{2(j-1)-(n_{k+1}-1)}\widehat{V}_k^{(0)}(Z) + Z^{n_{k+1}-2i-1}\widehat{V}_k^{(1)}(Z)
\end{aligned}$$

In case $\epsilon_k(0) = 1$, we have $T_{k+1} - T_k = T_{k+1} + T_k + 2s_k(0) = 1$.

- in this case, for $n_{k+1} - 1 - i \in 2\mathbb{Z}$,

$$
\begin{aligned}
&\widehat{V}_{k+1}^{(i)}(Z) = Z^{T_{k+1}}\widehat{U}_{k+1}^{(i)}(Z^2) \\
&= \frac{n_{k+1}-1-i}{2}\big(Z^{T_{k+1}}\widehat{U}_k^{(0)}(Z^2) + Z^{2s_k(0)+T_{k+1}}\widehat{U}_k^{(0)}(Z^{-2})\big) + Z^{T_{k+1}}\widehat{U}_k^{(1)}(Z^2) \\
&= \frac{n_{k+1}-1-i}{2}\big(Z^{T_{k+1}-T_k}\widehat{V}_k^{(0)}(Z) + Z^{2s_k(0)+T_{k+1}+T_k}\widehat{V}_k^{(0)}(Z^{-1})\big) + Z^{T_{k+1}-T_k}\widehat{V}_k^{(1)}(Z) \\
&= \frac{n_{k+1}-1-i}{2}\big(Z\widehat{V}_k^{(0)}(Z) + Z\widehat{V}_k^{(0)}(Z^{-1})\big) + Z\widehat{V}_k^{(1)}(Z)
\end{aligned}
$$

- in this case, for $n_{k+1} - 1 - i \in 2\mathbb{Z} + 1$,

$$
\begin{aligned}
&\widehat{V}_{k+1}^{(i)}(Z) = Z^{T_{k+1}}\widehat{U}_{k+1}^{(i)}(Z^2) \\
&= \frac{n_{k+1}-i}{2}Z^{T_{k+1}}\widehat{U}_k^{(0)}(Z^2) + \frac{n_{k+1}-2-i}{2}Z^{2s_k(0)+T_{k+1}}\widehat{U}_k^{(0)}(Z^{-2}) + Z^{2s_k(0)+T_{k+1}}\widehat{U}_k^{(1)}(Z^{-2}) \\
&= \frac{n_{k+1}-i}{2}Z^{T_{k+1}-T_k}\widehat{V}_k^{(0)}(Z) + \frac{n_{k+1}-2-i}{2}Z^{2s_k(0)+T_{k+1}+T_k}\widehat{V}_k^{(0)}(Z^{-1}) + Z^{2s_k(0)+T_{k+1}+T_k}\widehat{V}_k^{(1)}(Z^{-1}) \\
&= \frac{n_{k+1}-i}{2}Z\widehat{V}_k^{(0)}(Z) + \frac{n_{k+1}-2-i}{2}Z\widehat{V}_k^{(0)}(Z^{-1}) + Z\widehat{V}_k^{(1)}(Z^{-1})
\end{aligned}
$$

- and $\widehat{V}_{k+1}^{(1)}(Z) = Z\widehat{V}_k^{(1)}(Z)$ in case $n_{k+1} = 2$.

§4 Extracting the ARW

Associated sequence of temporal probabilities.

Now let $P_k^{(i)} \in \mathcal{P}(\mathbb{Z})$ be defined by

$$P_k^{(i)}(\nu) := \frac{V_k^{(i)}(\nu)}{\ell_k(i)}.$$

We call these "temporal probabilities" because

$$P_k^{(0)}(\nu) = \frac{1}{\ell_k(0)}\#\{1 \le j \le \ell_k(0) :\ 2\varphi_j(0) + T_k = \nu\}.$$

The *generating function* of $P_k^{(i)}$ is $\Phi_k^{(i)} : \mathbb{S}^1 \to \mathbb{C}$ defined by

$$\Phi_k^{(i)}(Z) := \sum_{\nu\in\mathbb{Z}} P_k^{(i)}(\nu)Z^\nu = \frac{\widehat{V}_k^{(i)}(Z)}{\ell_k(i)} \quad (Z \in \mathbb{S}^1).$$

Set

$$\phi_N(Z) := \frac{1}{N}\sum_{k=0}^{N-1} Z^k \quad (N \ge 1) \text{ and } \phi_0 = \phi_{-1} \equiv 0;$$

$$p_{k+1}(i) := 1 - \frac{\ell_k(1)}{\ell_{k+1}(i)} = \frac{(n_{k+1}-1-i)\ell_k(0)}{\ell_{k+1}(i)};$$

$$q_N := 1 - \frac{\lfloor \frac{N}{2} \rfloor}{N}, \quad (N \ge 1) \text{ and } q_0 := 0.$$

Generating function transitions.

For $k \ge 1$, $i = 0, 1$ and $Z \in \mathbb{S}^1$ we have,

- in case $\epsilon_k(0) = 0$,

$$\begin{aligned}\Phi_{k+1}^{(i)}(Z) &= \frac{\ell_k(0)}{\ell_{k+1}(i)}\sum_{j=1}^{n_{k+1}-1-i} Z^{2(j-1)-(n_{k+1}-1)}\Phi_k^{(0)}(Z) + \frac{1}{\ell_k(i)}Z^{n_{k+1}-2i-1}\Phi_k^{(1)}(Z)\\ &= p_{k+1}(i)Z^{-(n_{k+1}-1)}\phi_{n_{k+1}-1-i}(Z^2)\Phi_k^{(0)}(Z) + (1-p_{k+1}(i))Z^{n_{k+1}-2i-1}\Phi_k^{(1)}(Z)\end{aligned}$$

- in case $\epsilon_k(0) = 1$ and $2 \mid n_{k+1} - 1 - i$,

$$\begin{aligned}&\Phi_{k+1}^{(i)}(Z)\\ &= \frac{(n_{k+1}-1-i)\ell_k(0)}{2\ell_{k+1}(i)}(Z\Phi_k^{(0)}(Z) + Z\Phi_k^{(0)}(Z^{-1})) + \frac{1}{\ell_k(i)}Z\Phi_k^{(1)}(Z)\\ &= \frac{p_{k+1}(i)}{2}(Z\Phi_k^{(0)}(Z) + Z\Phi_k^{(0)}(Z^{-1})) + (1-p_{k+1}(i))Z\Phi_k^{(1)}(Z)\end{aligned}$$

- in case $\epsilon_k(0) = 1$ and $2 \nmid n_{k+1} - 1 - i$,

$$\begin{aligned}&\Phi_{k+1}^{(i)}(Z)\\ &= \frac{(n_{k+1}-i)\ell_k(0)}{2\ell_{k+1}(i)}Z\Phi_k^{(0)}(Z) + \frac{(n_{k+1}-2-i)\ell_k(0)}{2\ell_{k+1}(i)}Z\Phi_k^{(0)}(Z^{-1}) + \frac{1}{\ell_k(i)}Z\Phi_k^{(1)}(Z^{-1})\\ &= p_{k+1}(i)q_{n_{k+1}-i-1}Z\Phi_k^{(0)}(Z) + p_{k+1}(i)(1-q_{n_{k+1}-i-1})Z\Phi_k^{(0)}(Z^{-1}) + (1-p_{k+1}(i))Z\Phi_k^{(1)}(Z^{-1})\end{aligned}$$

Smelling the `RAT`s.

Next, for given α with $2\alpha = [n_1, n_2, \dots]$ we'll construct `ARW`s

$$X^{(k)} = (X^{(k)}(0), X^{(k)}(1)) \quad (k \ge 1)$$

so that

$$(☣) \qquad P([X^{(k)}(i) = \nu]) = P_k^{(i)}(\nu).$$

We call an `ARW` satisfying (☣) an α-`ARW`.

Let

$$(\clubsuit)\qquad \begin{aligned}&\mathcal{N}_N \in \mathtt{RV}(\mathbb{Z}),\quad P(\mathcal{N}_N = 2k-(N-1)) = \frac{1}{N},\ 0\le k\le N-1;\\ &x_k(i)\in \mathtt{RV}(\{0,1\}),\ P(x_k(i)=1)=p_k(i);\\ &y_N\in\mathtt{RV}(\{0,1\}),\ P(y_N=1)=q_N.\end{aligned}$$

Flip type RATs.

We call a $\mathtt{RAT}$ $F=(a,b)\in M_{d\times d}(\{-1,0,1\})\times\mathbb{R}^d$, *of flip type* if

$$a(k,L)=1_{[\mathfrak{L}(k,a)=L]}\widetilde{a}(k,L)$$

where each $\mathfrak{L}(k,a)$ is a RV with values in $\{1,2,\dots,d\}$ and each $\widetilde{a}(k,L)$ is a RV with values in $\{-1,1\}$.

Flip type is preserved under composition. For $F'=(a',b')$ and $F=(a,b)$,

$$F'\circ F=(a',b')\circ(a,b)=(a'a,a'b+b')$$

and

$$\mathfrak{L}(k,a'a)=\mathfrak{L}(\mathfrak{L}(k,a'),a).$$

The α-$\mathtt{ARW}$s to be constructed will be generated by $\mathtt{RAT}$s of flip type. Indeed, henceforward, we only consider flip type $\mathtt{RAT}$s

Linear recursion for characteristic functions of flip type ARWs.

Let $((a^{(n)},b^{(n)}):\ n\ge 1)$ be a flip type $\mathtt{RAT}$ sequence and consider the generated $\mathtt{ARW}$

$$X^{(0)}=0,\ X^{(n)}=a^{(n)}X^{(n-1)}+b^{(n)}.$$

The characteristic functions of the coordinates of $(X^{(n)},-X^{(n)})$ satisfy a linear recursion.

In the special case where $P([a_{k,\ell}^{(n)}=-1])=0\ \forall\ n\ge 1,\ 1\le k,\ell\le d$, there is a simpler linear recursion for the characteristic functions of the coordinates of $X^{(n)}$.

Writing for the $\mathbb{R}^d$-valued random variable $X=(X_1,\dots,X_d)$:

$$V_X(\theta):=\begin{pmatrix}\widehat{\Phi}_{X_1}(\theta)\\ \vdots\\ \widehat{\Phi}_{X_d}(\theta)\\ \widehat{\Phi}_{X_1}(-\theta)\\ \vdots\\ \widehat{\Phi}_{X_d}(-\theta)\end{pmatrix},$$

where $\widehat{\Phi}_Y(\theta):=E(e^{i\theta Y})$ denotes the characteristic function of the $\mathbb{R}$-valued random variable Y;

and for the flip type RAT (a, b) independent of X :

$$X' = aX + b,$$

we have that

$$V_{X'}(\theta) = P(\theta)V_X(\theta).$$

Here $P(\theta) \in M_{2d\times 2d}(\mathbb{C})$ is given by

$$P_{k,L}(\theta) = P_{k,L}(0)\widehat{\Phi}_{C_{k,L}}(\theta)$$

where $P(0)$ is a stochastic matrix and $C_{k,L}$ $(1 \le k, L \le 2d)$ are random variables.

Specifically:

$$P_{k,L}(0) = P([\mathfrak{L}(k,a) = L])P([\widetilde{a}_{k,L} = 1]) \text{ and } C_{k,L} = b_{k,L,1};$$
$$P_{k,d+L}(0) = P([\mathfrak{L}(k,a) = L])P([\widetilde{a}_{k,L} = -1]) \text{ and } C_{k,d+L} = b_{k,L,-1};$$
$$P_{d+k,L}(0) = P([\mathfrak{L}(k,a) = L])P([\widetilde{a}_{k,L} = -1]) \text{ and } C_{d+k,L} = -b_{k,L,-1};$$
$$P_{d+k,d+L}(0) = P([\mathfrak{L}(k,a) = L])P([\widetilde{a}_{k,L} = 1]) \text{ and } C_{d+k,d+L} = -b_{k,L,1}.$$

Here, for $1 \le k, L \le d$, $J \in \mathbb{Z}$ and for $\epsilon = \pm 1$,

$$P([b_{k,L,\epsilon} = J]) := P([b_k = J] \| [a_{k,L} = \epsilon]).$$

Equivalently, for $i, j = 0, 1$ and $\epsilon = 1 - 2i$, $\delta = 1 - 2j$

$$P_{id+k,jd+L}(\theta) = P([\mathfrak{L}(k,a) = L])P([\widetilde{a}_{k,L} = \delta])\widehat{\Phi}_{\epsilon\delta b_{k,\ell,\delta}}(\theta).$$

We'll refer to the function $P : \mathbb{R} \to M_{2d\times 2d}(\mathbb{C})$ as the *characteristic function* of the flip type RAT: (abbr. RAT-CF).

Construction procedures.

Fix a sequence of independent random vectors

$$(x_{k+1}(i), y_{n_{k+1}-1-i}, \mathcal{N}_{n_{k+1}-1-i}), \quad i = 0,1)_{k\ge 1}$$

whose marginals are determined by $2\alpha = [n_1, n_2, \dots]$ and (☢).

Define $X^{(k)} = (X^{(k)}(0), X^{(k)}(1)) \in \mathrm{RV}(\mathbb{Z}^2)$ by $X^{(0)}(i) = 0, \quad i = 0,1$ and

- in case $\epsilon_k(0) = 0$,

$$X^{(k+1)}(i) = x_k(i)X^{(k)}(0) + (1-x_k(i))X^{(k)}(1) + x_k(i)\mathcal{N}_{n_{k+1}-1-i} + (1-x_k(i))(n_{k+1}-2i-1);$$

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z}$,

$$X^{(k+1)}(i) = x_k(i)(2y_{n_{k+1}-i} - 1)X^{(k)}(0) + (1 - x_k(i))X^{(k)}(1) + 1$$

- in case $\epsilon_k(0) = 1$ and $n_{k+1} - 1 - i \in 2\mathbb{Z} + 1$,

$$X^{(k+1)}(i) = x_k(i)(2y_{n_{k+1}-i} - 1)X^{(k)}(0) - (1 - x_k(i))X^{(k)}(1) + 1.$$

The form of the RATs and the independence of the random vectors implies that $(X^{(k)}:\ k\geq 0)$ is indeed a flip type ARW.

Construction Lemma 4.1 *A flip type* ARW *is an* α-ARW *iff it is defined according to a construction procedure as above .*

Proof Evidently, (☣) holds if and only if the ARW generating function transitions are the same as those established above for the $(\Phi_k^{(i)}(Z):\ k\geq 0,\ i=0,1)$. The latter correspond to linear recursions by RAT-CFs of flip type RATs as above. ☑

Remark. The plethora of α-ARWs arises because of the variety of possible joint distributions of the random vectors

$$\{(x_{k+1}(i), y_{n_{k+1}-1-i}, \mathcal{N}_{n_{k+1}-1-i}):\quad i=0,1\}$$

for fixed $k\geq 1$.

α-RAT sequences.

An α-RAT *sequence* is a flip type RAT sequence $(F_k=(a^{(k)},b^{(k)}):\ k\geq 1)$ which generates an α-ARW $(X^{(k)}:\ k\geq 0)$ by

$$X^{(0)}=0,\ X^{(k)}=a^{(k)}X^{(k-1)}+b^{(k)}.$$

The α-RAT sequences $(F_k=(a^{(k)},b^{(k)}):\ k\geq 1)$ satisfy, for $n_k\geq 3$,

(0)
$$a^{(k)}=\begin{pmatrix} x_k(0) & 1-x_k(0)\\ x_k(1) & 1-x_k(1)\end{pmatrix},\qquad \epsilon_{k-1}=0;$$

(1)
$$a^{(k)}=\begin{pmatrix} x_k(0)(2y_{n_k}-1) & -(1-x_k(0))\\ x_k(1)(2y_{n_k-1}-1) & (1-x_k(1))\end{pmatrix}\qquad \epsilon_{k-1}=1 \text{ and } 2\mid n_k;$$

(1)
$$a^{(k)}=\begin{pmatrix} x_k(0)(2y_{n_k}-1) & (1-x_k(0))\\ x_k(1)(2y_{n_k-1}-1) & -(1-x_k(1))\end{pmatrix}\qquad \epsilon_{k-1}=1 \text{ and } 2\nmid n_k$$

(0) $$b^{(k)}=\begin{pmatrix} x_k(0)\mathcal{N}_{n_k-1}+(1-x_k(0))(n_k-1)\\ x_k(1)\mathcal{N}_{n_k-2}+(1-x_k(1))(n_k-2)\end{pmatrix}\qquad \epsilon_{k-1}=0;$$

(1) $$b^{(k)}=\begin{pmatrix}1\\1\end{pmatrix}\qquad \epsilon_{k-1}=1$$

and for $n_{k+1} = 2$ by

(0) $$a^{(k)} = \begin{pmatrix} x_k(0) & 1 - x_k(0) \\ 0 & 1 \end{pmatrix}, \qquad \epsilon_{k-1} = 0;$$

(1) $$a^{(k)} = \begin{pmatrix} x_k(0)(2y_{n_k} - 1) & -(1 - x_k(0)) \\ 0 & 1 \end{pmatrix} \qquad \epsilon_{k-1} = 1$$

(0) $$b^{(k)} = \begin{pmatrix} 1 - x_k(0) \\ 0 \end{pmatrix} \qquad \epsilon_{k-1} = 0;$$

(1) $$b^{(k)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \epsilon_{k-1} = 1.$$

Special `RAT`s. We call `RAT`s of the type defined above *special* (`spec-RAT`s).

We define the *parity* of a `spec-RAT` F as above to be *even* if it is defined by equations marked "(0)", and to be *odd* if it is defined by equations marked "(1)".

The *coefficient* associated to the `spec-RAT` F_k as above is $n(F_k) := n_k$.

A `spec-RAT` with coefficient 2 is called *trivial*.

§5 Asymptotics of `ARW`s

Norm of a matrix Throughout this paper, we use the L^∞-operator norm of matrices. Namely, the norm of the matrix $A \in M_{d\times d}(\mathbb{C}) \cong \mathbb{C}^{d^2}$ is

$$\|A\| := \sup\{\|Ax\|_\infty : \ x \in \mathbb{C}^d, \ \|x\|_\infty = 1\} = \max_k \sum_{\ell=1}^d |A_{k,\ell}|$$

where $\|(x_1, x_2, \dots, x_d)\|_\infty := \max_{1\le k\le d} |x_k|$.

Norm of a `RAT-CF`.

Let $P = P_F : \mathbb{T} \to M_{2d\times 2d}(\mathbb{C})$ be the characteristic function of the `RAT` $F = (a, b)$, then for $\theta \in \mathbb{T}$,

(🦇) $$\begin{aligned} \|P(\theta)\| &= \max_{1\le J\le 2d} \sum_{K=1}^{2d} |P_{J,K}(\theta)| \\ &= \max_{1\le k\le d} \sum_{L=1}^{d} \sum_{L=1}^{d} [P(a_{k,L} = 1)|\widehat{\Phi}_{b_{k,L,1}}(\theta)| + P(a_{k,L} = -1)|\widehat{\Phi}_{b_{k,L,-1}}(\theta)|] \\ &\le 1 \end{aligned}$$

with equality iff for some k, $b_{k,\ell,\epsilon}$ is a constant random variable $\forall\ 1 \le \ell \le d$ and $\epsilon = \pm 1$ with $P(a_{k,\ell} = \epsilon) > 0$.

Irreducibility, mean contractivity and balance.

We call the `RAT` F *irreducible* if $P(a_{k,\ell}(F) \neq 0) > 0 \ \forall\ 1 \le k, \ell \le d$ and *mean contractive* if $\|E(a(F))\| < 1$. Note that F is mean contractive iff for each $1 \le k \le d\ \exists\ 1 \le \mu \neq \nu \le d$ so that $P(a_{k,\mu} \neq 0) > 0$ and $P(a_{k,\nu} = \epsilon) > 0\ \forall\ \epsilon = \pm 1$.

For `spec-RAT`s, mean contractivity entails irreducibility.

Nontrivial `spec-RAT`s with odd parity are irreducible and mean contractive.

Nontrivial `spec-RAT`s with even parity are irreducible but not mean contractive.

Trivial `spec-RAT`s are not irreducible.

Call F *balanced* if $P(a_{k,\ell} = \epsilon) > 0\ \forall\ 1 \le k, \ell \le d$ and $\epsilon = \pm 1$. Balance entails both irreducibility and mean contractivity.

If either of F', F is irreducible, mean contractive or balanced, then so is $F' \circ F$.

If F is irreducible and F' is mean contractive, then $F' \circ F$ is balanced (but $F \circ F'$ may not be balanced)

Adapted `RAT`s.

We'll call the `RAT` $(a, b) \in M_{d\times d}(\mathbb{R}) \times \mathbb{R}^d$

- *adapted* if $\exists\ \theta \in \mathbb{T}$ so that $\|P_{(a,b)}(\theta)\| < 1$;
- *strongly adapted* if $\|P_{(a,b)}(\theta)\| < 1\ \forall\ \theta \neq 0$;
- *partially adapted* if a has an *adapted row* i.e.: $\exists\ 1 \le k \le d$ and $\theta \in \mathbb{T}$ so that

$$\sum_{L=1}^{d} [P(a_{k,L} = 1)|\widehat{\Phi}_{b_{k,L,1}}(\theta)| + P(a_{k,L} = -1)|\widehat{\Phi}_{b_{k,L,-1}}(\theta)|] < 1;$$

equivalently $\exists\ 1 \le \ell \le d$ and $\epsilon = \pm 1$ with $P(a_{k,\ell} = \epsilon) > 0$, $b_{k,\ell,\epsilon}$ is a non-constant random variable.

Note that the `RAT` is adapted iff all rows are adapted.

The `RAT` $(a, b) \in M_{d\times d}(\mathbb{R}) \times \mathbb{R}^d$ with $E(\|b\|^2) < \infty$ is adapted iff

$$\begin{aligned}\kappa_F &:= \min_k - \sum_{\ell=1}^{2d} \frac{d^2}{d\theta^2} P_F(\theta)_{k,\ell}|_{\theta=0} \\ &= \min_k \sum_{L=1}^{d} P(\mathfrak{L}(k,a) = L)[P(\widetilde{a}_{k,L} = 1)\mathtt{Var}(b_{k,L,1}) + P(\widetilde{a}_{k,L} = -1)\mathtt{Var}(b_{k,L,-1})] > 0.\end{aligned}$$

The *periodicity group* of the `RAT` $F=(a,b)$ is

$$\Gamma_F := \{\theta\in\mathbb{R}:\ \|P_F(\theta)\|=1\}.$$

It follows from (🦇) that

☼
$$\Gamma_F=\bigcup_{k=1}^{d}\bigcap_{1\le\ell\le d}\ \bigcap_{\epsilon=\pm1,P(a_{k,\ell}=\epsilon)>0}\Gamma_{b_{k,\ell,\epsilon}}$$

where for a random variable b,

$$\Gamma_b := \{\theta\in\mathbb{R}:\ |\widehat{\Phi}_b(\theta)|=1\}.$$

This is a closed subgroup of $\mathbb{R}$ and so, either $\Gamma_F=\mathbb{R}$ or F is adapted and $\Gamma_F=g\mathbb{Z}$ for some $g=g_F>0$.

Note that F is strongly adapted iff $\Gamma_F=\{0\}$.

In case F is discrete, adapted, $g_F=\frac{2\pi}{N}$ for some $N\ge1$ and $\Gamma_F=\{\frac{2\pi k}{N}:\ 0\le k\le N-1\}$ is a finite subgroup of $\mathbb{T}$. Moreover

$$\|P_F(\theta+\gamma)\|=\|P_F(\theta)\|\ \forall\ \theta\in\mathbb{T}\text{ and }\gamma\in\Gamma_F$$

and $\forall\ 0<\epsilon<\frac{1}{2|\Gamma_F|}\ \exists\ \Delta>0$ so that

$$\|P_F(\theta)\|\le1-\Delta\ \forall\ \theta\in\mathbb{T}\setminus B(\Gamma_F,\epsilon).$$

Here, for (X,d) a metric space, $\Gamma\subset X$ and $\epsilon>0$,

$$B(\Gamma,\epsilon):=\{x\in X:\ \exists\ y\in\Gamma,\ d(x,y)\le\epsilon\}.$$

Adapted `spec-RAT`s Any nontrivial `spec-RAT` F with even parity is:
- adapted if $n(F)\ge4$ and
- partially adapted if $n(F)=3$.

No trivial `spec-RAT` or `spec-RAT` with odd parity is partially adapted.

Spectral properties of `RAT-CF`s.

Let F be a `RAT`. The `RAT-CF` $P_F:\mathbb{R}\to M_{2d\times2d}(\mathbb{C})$ is a *symmetric function* in the sense that $P_F(-\theta)=\overline{P_F(\theta)}$.

Noting that

$$P_F(\theta)=\begin{pmatrix}Q(\theta)&R(\theta)\\R(-\theta)&Q(-\theta)\end{pmatrix}=\begin{pmatrix}Q(\theta)&R(\theta)\\\overline{R}(\theta)&\overline{Q}(\theta)\end{pmatrix}$$

where

$$Q(\theta)=\sum_\nu Q_\nu e^{i\nu\theta}\ \text{ and }\ R(\theta)=\sum_\nu R_\nu e^{i\nu\theta}$$

with each $Q_\nu,\ R_\nu\in M_{d\times d}(\mathbb{C})$.

By applying the row and column permutations

$$(1,2,\dots,2d)\mapsto(d+1,\dots,2d,1,2,\dots,d)$$

(as in [3]), we see that for $\lambda \in \mathbb{C}$,

$$\det(P_F(\theta) - \lambda I) = \det(P_F(-\theta) - \lambda I) = \det(\overline{P}_F(\theta) - \lambda I).$$

It follows that

$$(⚽) \qquad \det(P_F(\theta) - \lambda I) = \sum_{k=0}^{2d} c_k(\theta)\lambda^k \quad \text{where} \quad c_k : \mathbb{T} \to \mathbb{R} \text{ is even.}$$

Spectral theory of RAT-CFs.

As seen above a RAT-CF is a perturbation of a positive operator. The spectral theory of perturbations (as in chapter III of [12]) is applicable.

Let F be a RAT with $E(\|b(F)\|^2) < \infty$. We call $P_F(0)$ *positive* if $P(0)_{k,\ell} > 0 \ \forall \ 1 \le k, \ell \le d$ (equivalently, F is balanced), and *aperiodic* if $\exists \ N \ge 1$ with $P_F(0)^N$ is positive.

Suppose that $P_F(0)$ is aperiodic.

By the Perron-Frobenius theorem, $1 = \|P_F(0)\|$ is a simple, dominant eigenvalue (i.e. its eigenspace $\mathbb{C} \cdot 1$ is one-dimensional and all other eigenvalues are smaller in absolute value).

Since $\theta \mapsto P_F(\theta)$ is C^2, by the implicit function theorem, $\exists \ \epsilon > 0$ so that for $|\theta| < \epsilon$, $P_F(\theta)$ also has a simple, dominant eigenvalue $\lambda_F(\theta) \in \mathbb{C}$ where $\lambda_F : (-\epsilon, \epsilon) \to \mathbb{R}$ is differentiable.

By (⚽) $\lambda_F : (-\epsilon, \epsilon) \to \mathbb{R}$ is a real valued, even function with $\lambda_F'(0) = 0$ and $\gamma_F := -\lambda_F''(0) \ge 0$ with equality iff $\lambda_F \equiv 1$.

If F is adapted, then $\gamma_F > 0$. However, it may be that $\gamma_F > 0$ and $\|P_F\| \equiv 1$.

Fix $\pi = \pi_F \in \mathbb{R}_+^d$ satisfying

$$\langle \pi, 1 \rangle = 1 \text{ and } P_F(0)^t \pi = \pi.$$

There is a unique eigenvalue function $\eta = \eta_F : (-\epsilon, \epsilon) \to \mathbb{C}^{2d}$ so that

$$P_F(\theta)\eta(\theta) = \lambda(\theta)\eta(\theta) \text{ and } \langle \pi, \eta(\theta) \rangle = 1 \quad \forall \ |\theta| < \epsilon.$$

It follows that $\eta(0) = 1$ and $\eta(-\theta) = \overline{\eta(\theta)}$.

Theorem 5.1 Coordinate distributional limits for stationary `ARW`s

Let $F_n = (a_n, b_n) \in M_{d\times d}(\mathbb{R}) \times \mathbb{R}^d \quad (n \geq 1)$ be an iid `RAT` *sequence with each $F_n \overset{dist}{=} F$ with F adapted and $P_F(0)$ aperiodic and let $(X_1, X_2, \dots)$ be the associated* `ARW`*, then for each $1 \leq k \leq d$*

$$\text{(CLT)} \qquad \frac{(X_n)_k}{\sqrt{n}} \xrightarrow[n\to\infty]{\texttt{dist}} \mathcal{N}(0, \gamma_F)$$

If, in addition, F is strongly adapted, then for any bounded interval $J \subset \mathbb{R}$,

$$\text{(LLT)} \qquad \sqrt{n}P([(X_n)_k \in t_n + J]) \xrightarrow[n\to\infty,\ \frac{t_n}{\sqrt{n}}\to x]{} \frac{1}{2\pi\gamma_F} e^{-\frac{x^2}{2\gamma_F}}.$$

Proof sketch Let $P = P_F$ and let $\pi \in \mathbb{R}_+^{2d}$ be so that $P_F(0)^t\pi = \pi$. For each $\theta \in (-\epsilon, \epsilon)$, $\exists\,!\ \eta(\theta) \in \mathbb{C}^d$ so that

$$P_F(\theta)\eta(\theta) = \lambda_i(\theta)\eta(\theta) \text{ and } \langle \eta(\theta), \pi\rangle = 1.$$

It follows that $\eta : (-\epsilon, \epsilon) \to \mathbb{C}^d$ is a smooth, symmetric function.

Again, for each $\theta \in (-\epsilon, \epsilon)$, $\exists\,!\ \pi(\theta) \in \mathbb{C}^d$ so that for $\theta \in (-\epsilon, \epsilon)$,

$$P_F(\theta)\pi(\theta) = \lambda(\theta)\pi(\theta) \text{ and } \langle \pi(\theta), \eta(\theta)\rangle = 1.$$

It follows that $\pi : (-\epsilon, \epsilon) \to \mathbb{C}^d$ is also a smooth symmetric function.

Define the projection $N(\theta) : \mathbb{C}^d \to \mathbb{C}^d$ by

$$N(\theta)x := \langle x, \pi(\theta)\rangle\eta(\theta).$$

We see that

$$P_F(\theta)N(\theta) = \lambda(\theta)N(\theta),$$

and that $Q(\theta) := P_F(\theta) - P_F(\theta)N(\theta) : \mathbb{C}^d \to \mathbb{C}^d$ satisfies $QN = NQ = 0$. The spectral radius of $Q(0)$, $r(Q(0)) < 1$. By continuity of $\theta \mapsto Q(\theta)$, for possibly smaller $\epsilon > 0$,

$$r(Q(\theta)) \leq \rho < 1\ \forall\ |\theta| < \epsilon.$$

Thus

$$V_{X_n}(\theta) = P_F(\theta)^n 1 = \lambda(\theta)^n\eta(\theta) + O(\rho^n).$$

The first statement follows directly from this.

The second follows also since if F is strongly adapted, $\sup_{|\theta|\in[\epsilon,M]} |\lambda(\theta)| < 1\ \forall\ 0 < \epsilon < M < \infty$. This is seen via standard proofs of the local limit theorem (see [6]). ☑

Remark: Irreducible, positive `RAT`s

The above does not include the important case of an irreducible `RAT` (a, b) with $P(a_{l,\ell} = -1) = 0\ \forall\ k,\ \ell$. Here there is possible linear drift,

but the d-dimensional vectors $(\widehat{\Phi}_{X_1^{(n)}}, \widehat{\Phi}_{X_2^{(n)}}, \dots, \widehat{\Phi}_{X_d^{(n)}})$ satisfy a linear recursion and an analogous spectral argument applies.

The following is a special case of a subsequence version of theorem 1.1 in [5] (see also [4]).

Corollary 5.2 *Suppose that $\alpha \in (0,1)$ is quadratic with $2\alpha = [\overline{m_1, m_2, \dots, m_r}]$. There are constants $c > 0$ and $\mu \in \mathbb{R}$ so that for intervals $I \subset \mathbb{R}$*

$$\frac{1}{\ell_{rk}(0)} \# \left\{ 1 \le j \le \ell_{rk}(0) : \ \frac{\varphi_j^{(\alpha)}(0) - \mu k}{c\sqrt{k}} \in \ I \right\} \xrightarrow[k\to\infty]{} \frac{1}{2\pi} \int_I e^{\frac{-t^2}{2}} dt.$$

Here, $\varphi_j^{(\alpha)}(0) := \sum_{t=0}^{j-1} \varphi \circ r_\alpha(0)$.

Proof The periodicity of the expansion of 2α implies that the α-`RAT` sequence is eventually periodic and the above coordinate `CLT` in 5.1 can be applied. ☑

Remark: Quadratic α

The corollary remains true for general quadratic α. The expansion of 2α is now only eventually periodic but α-`RAT` sequences may not be. However as in [3], the transitions of the generating functions of the simplified visit distributions are eventually periodic and the result can be deduced from this as was (🚗) (see page 2) in this case.

In general, for $\alpha \in$ `BAD` there need be no eventual periodicity of the `RAT` sequence (or related objects) and new tools are needed.

§6 The weak, rough local limit theorem

Variance of `ARW`s.

We call the `RAT` (a,b) *centered* if $E(b) = 0$ and the `RAT` sequence $(F_n = (a^{(n)}, b^{(n)}) : \ n \ge 1)$ *centered* if each `RAT` $F_n = (a^{(n)}, b^{(n)})$ is centered.

From the formula

$$X^{(n)} = F_n(X^{(n-1)}) = a^{(n)} X^{(n-1)} + b^{(n)}$$

we see that

$$E(X^{(n)}) = E(a^{(n)}) E(X^{(n-1)}) + E(b^{(n)})$$

whence the `RAT` sequence $(F_n = (a^{(n)}, b^{(n)}) : \ n \ge 1)$ is centered if and only if

$$E(X^{(n)}) = 0 \ \forall \ n \ge 1.$$

6.1 Variance Theorem

Let $(F_n := (a^{(n)}, b^{(n)}) : \ n \geq 1)$ be a centered, independent `RAT` *sequence and let*

$$X^{(n)} := F_1^n(0) = F_n \circ F_{n-1} \cdots \circ F_1(0)$$

be the corresponding `ARW`*, then for $1 \leq k \leq d$ and $n \geq 1$,*

$$\sum_{\nu=1}^{n} \min_{1 \leq L \leq d} E([b_L^{(\nu)}]^2) \leq E([X_k^{(n)}]^2) \leq \sum_{\nu=1}^{n} \max_{1 \leq L \leq d} E([b_L^{(\nu)}]^2).$$

Proof

Set

$$Y^{(n,\nu)} := a_{\nu+1}^n b^{(\nu)} \quad (1 \leq \nu \leq n)$$

where

$$a_k^N := \begin{cases} a^{(N)} a^{(N-1)} \cdots a^{(k)} & k \leq N \\ \mathrm{Id} & k > N. \end{cases}$$

so that

$$X^{(n)} = \sum_{\nu=1}^{n} Y^{(n,\nu)}.$$

As above, $E(Y_K^{(n,\nu)}) = 0$.

¶ for each $1 \leq K \leq d, \ n \geq 1$, the random variables $\{Y_K^{(n,\nu)} : \ 1 \leq \nu \leq n\}$ are orthogonal, i.e.

$$E(Y_K^{(n,\nu)} Y_K^{(n,\mu)}) = 0 \ \forall \ \mu \neq \nu.$$

Proof of ¶

We have for $1 \leq K \leq d$, that

$$(\text{✐}) \qquad Y_K^{(n,\nu)} = (a_k^N)_{K,\ell(K,a_{\nu+1}^n)} b_{\ell(K,a_{\nu+1}^n)}^{(\nu)}$$

and for $1 \leq \nu < \mu \leq n$, that

$\ell(K, a_{\nu+1}^n) = \ell(\ell(K, a_{\mu+1}^n), a_{\nu+1}^\mu)$ and

$(a_{\nu+1}^n)_{K,\ell(K,a_{\nu+1}^n)} = (a_{\mu+1}^n)_{K,\ell(K,a_{\mu+1}^n)} (a_{\nu+1}^\mu)_{\ell(K,a_{\mu+1}^n),\ell(\ell(K,a_{\mu+1}^n),a_{\nu+1}^\mu)}$

Consequently

$$
\begin{aligned}
&Y_K^{(n,\nu)}Y_K^{(n,\mu)}\\
&\quad= (a_{\nu+1}^n)_{K,\ell(K,a_{\nu+1}^n)}b^{(\nu)}_{\ell(K,a_{\nu+1}^n)}(a_{\mu+1}^n)_{K,\ell(K,a_{\mu+1}^n)}b^{(\mu)}_{\ell(K,a_{\mu+1}^n)}\\
&\quad= (a_{\nu+1}^\mu)_{\ell(K,a_{\mu+1}^n),\ell(\ell(K,a_{\mu+1}^n),a_{\nu+1}^\mu)}b^{(\nu)}(\ell(\ell(K,a_{\mu+1}^n),a_{\nu+1}^\mu))b^{(\mu)}_{\ell(K,a_{\mu+1}^n)}\\
&\quad= \sum_{L=1}^d 1_{[\ell(K,a_{\mu+1}^n)=L]}b_L^{(\mu)}\sum_{M=1}^d 1_{[\ell(L,a_{\nu+1}^\mu)=M]}(a_{\nu+1}^\mu)_{L,M}b_M^{(\nu)}
\end{aligned}
$$

whence by independence and centering,

$$
\begin{aligned}
&E(Y^{(n,\nu)}{}_K Y_K^{(n,\mu)})\\
&\quad= \sum_{L=1}^d P([\ell(K,a_{\mu+1}^n)=L])E(b_L^{(\mu)})\sum_{M=1}^d E(1_{[\ell(L,a_{\nu+1}^\mu)=M]}(a_{\nu+1}^\mu)_{L,M})E(b_M^{(\nu)})\\
&\quad= 0. \quad ☑¶
\end{aligned}
$$

It follows from (✔) that

$$E(Y_K^{(n,\nu)2}) = \sum_{L=1}^d P([\ell(K,a_{\nu+1}^n)=L])E(b_L^{(\nu)2})$$

and from the above that

$$E(X_K^{(n)2}) = \sum_{\nu=1}^n E(Y_K^{(n,\nu)2})).$$

The Variance Theorem follows from this. ☑

Compactness properties of `RAT` sequences.

We'll say that the `RAT` sequence

$$(F_n = (a^{(n)},b^{(n)}):\ n\ge 1)\in \mathtt{RV}(M_{d\times d}(\mathbb{R})\times\mathbb{R}^d)^{\mathbb{N}}$$

- is *adapted* if $\exists$ a finite subgroup $\Gamma\le\mathbb{T}$ (called the *adaptivity group*) so that
$|\widehat{\Phi}_{b^{(n)}_{k,\ell,\pm1}}(\theta)| = 1 \implies \theta\in\Gamma$, and
$\forall\ \epsilon>0\ \exists\ \delta>0$ so that
$$|\widehat{\Phi}_{b^{(n)}_{k,\ell,\pm1}}(\theta)|\le 1-\delta\ \forall\ \ 1\le k,\ell\le d,\ n\ge 1\ \text{ and } \theta\in\mathbb{T}\smallsetminus B(\Gamma,\delta).$$

- is *uniform* if $\sup\{|E(b^{(n)}_{k,\ell,\epsilon})| :<\infty\ \forall\ \ 1\le k,\ell\le d,\ n\ge 1,\ \epsilon=\pm1$ and

(a) the collection

$$\mathcal{F}_F := \{|\overline{b}^{(n)}_{k,\ell,\epsilon}|^2 : \ \ 1 \le k, \ell \le d, \ n \ge 1, \ \epsilon = \pm 1\}$$

is uniformly integrable where $\overline{Y} := Y - E(Y)$ denotes the centering of the integrable random variable Y; and

(b) $\exists\ M > 1$ so that

$$\mathtt{Var}\,(b^{(n)}_{k,\ell,\epsilon}) = M^{\pm 1}\sum_{j=1}^{d}\mathtt{Var}\,(b^{(n)}_j)$$
$$\forall\ n \ge 1, \ 1 \le k, \ell \le d, \ \epsilon = \pm 1 \text{ with } \ P(a^{(n)}_{k,\ell} = \epsilon) > 0.$$

It is standard to show that uniformity of $((a^{(n)}, b^{(n)}) : \ n \ge 1)$ implies that $\exists\ M > 1$ so that $\mathtt{Var}\,(b^{(n)}_{k,\ell,\epsilon}) = M^{\pm 1} \ \ \forall\ n \ge 1, \ 1 \le k, \ell \le d, \ \epsilon = \pm 1$ with $P(a^{(n)}_{k,\ell} = \epsilon) > 0$.

The above properties do not entail discreteness and we'll need the WRLLT for certain non-discrete ARWs.

Theorem 6.2 (WRLLT) *Let* $(F_n = (a^{(n)}, b^{(n)}) : \ n \ge 1) \in (M_{d\times d}(\mathbb{R}) \times \mathbb{R}^d)^{\mathbb{N}}$ *be a* RAT *sequence.*

Suppose that $(F_n : \ n \ge 1)$ *is adapted, centered and uniform and let* $(X^{(n)})_{n\ge 1}$ *be the corresponding* ARW*.*

For each $1 \le k \le d$ *and* $1 \le p \le 2$*,*

$$\int_{\mathbb{T}} |\widehat{\Phi}_{X^{(n)}_k}(\theta)|^p d\theta \asymp \frac{1}{\sqrt{n}}.$$

Proof of the WRLLT Since the integral decreases with p, it suffices to show that $\exists\ M > 1$ so that $\forall\ n \in \mathbb{N}$ large,

$$\text{(a)} \quad \int_{\mathbb{T}} |\widehat{\Phi}_{X^{(n)}_k}(\theta)|^2 d\theta \ \ge \ \frac{1}{M\sqrt{n}} \quad \text{and} \quad \text{(b)} \quad \int_{\mathbb{T}} |\widehat{\Phi}_{X^{(n)}_k}(\theta)| d\theta \ \le \ \frac{M}{\sqrt{n}}.$$

Proof of (a) It follows from adaptedness, centeredness and the Variance theorem that $\exists\ \Gamma > 0$ so that

$$E(X^{(n)2}_k) \le \Gamma n \quad (1 \le k \le d).$$

Next, fix $M = 2\sqrt{\Gamma}$, then by Chebyshev's inequality,

$$P([|X^{(n)}_k| \le M\sqrt{n}]) \ge \frac{3}{4}.$$

Now fix $\Delta > 0$ so that

$$|1 - e^{it}| < \frac{1}{4} \ \forall \ |t| < \Delta.$$

We have

$$\begin{aligned}\sqrt{n}\int_{\mathbb{T}} |E(e^{i\theta X_k^{(n)}})|^2 d\theta &= \int_{-\pi\sqrt{n}}^{\pi\sqrt{n}} |E(\exp[it\frac{X_k^{(n)}}{\sqrt{n}})|^2 dt \\ &\geq \int_{-\frac{\Delta}{M}}^{\frac{\Delta}{M}} |E(\exp[it\frac{X_k^{(n)}}{\sqrt{n}}])|^2 dt.\end{aligned}$$

For $|t| < \frac{\Delta}{M}$, we have

$$\begin{aligned}|E(\exp[it\frac{X_k^{(n)}}{\sqrt{n}})| &\geq |E(\exp[it\frac{X_k^{(n)}}{\sqrt{n}}]1_{[|X_k^{(n)}|<M\sqrt{n}]})| - P([|X_k^{(n)}| \geq M\sqrt{n}] \\ &\geq \frac{3}{4}\cdot\frac{3}{4} - \frac{1}{4} \\ &= \frac{5}{16}\end{aligned}$$

whence

$$\begin{aligned}\sqrt{n}\int_{\mathbb{T}} |E(e^{i\theta X_k^{(n)}})|^2 d\theta &\geq \int_{-\frac{\Delta}{M}}^{\frac{\Delta}{M}} |E(\exp[it\frac{X_k^{(n)}}{\sqrt{n}}])|^2 dt \\ &\geq \frac{2\Delta}{M}\cdot\frac{25}{256}. \quad ☑\text{(a)}\end{aligned}$$

Proof of (b) Suppose that $\Gamma = \{\frac{k}{N} : \ 0 \leq k \leq N-1\} \subset \mathbb{T}$.
We claim first that

¶ $\exists \ \rho \in (0,1), \ 0 < \Delta < \frac{1}{2N}$ and $\beta > 0$ so that for each $n \geq 1$ and $\gamma \in \Gamma_{F_n}$,

(i) $\quad \|P_{F_n}(\gamma + \theta)\| \leq 1 - \beta\theta^2 \quad \forall \ |\theta| < \Delta;$ and

(ii) $\quad \|P_{F_n}(\gamma + \theta)\| \leq \rho \quad \forall \ \theta \notin B(\mathcal{F}, \Delta)$

(iii) $\quad \rho \leq 1 - \beta\Delta^2.$

Proof of ¶(i) Let Y be a random variable with finite second moment and let $W = Y - Y'$ be its symmetrization. We have for $\theta \in \mathbb{T} \cong (-\frac{1}{2}, \frac{1}{2})$ that

$$|\Phi_Y(\theta)|^2 = \Phi_W(\theta) = 1 - 2E(\sin^2(\tfrac{W\theta}{2})).$$

Now

$$\begin{aligned} E(\sin^2(\tfrac{W\theta}{2})) &\ge E(1_{[|W|\le\frac{\pi}{\theta}]}\sin^2(\tfrac{W\theta}{2})) \\ &\ge \frac{2\theta^2}{\pi^2}E(1_{[|W|\le\frac{\pi}{\theta}]}W^2) \\ &= \frac{2\theta^2}{\pi^2}E(W^2)(1-\Delta_Y(\theta)) \end{aligned}$$

where

$$\Delta_Y(\theta) := \frac{E(1_{[|W|>\frac{\pi}{\theta}]}W^2)}{E(W^2)}.$$

It follows from uniformity that $\exists\ \Delta > 0,\ \Delta < \frac{1}{2N}$ so that

$$\Delta_{\overline{b}^{(n)}_{k,\ell,\epsilon}} < \frac{1}{2}\quad \forall\ |\theta| < \Delta,\ 1 \le k,\ell \le d,\ n \ge 1$$

whence for $|\theta| < \Delta$ and $\gamma \in \Gamma_{F_n}$,

$$\begin{aligned} \|P_{F_n}(\gamma+\theta)\| &= \|P_{F_n}(\theta)\| \\ &= \max_{1\le k\le d}\sum_{L=1}^{d} P(\mathfrak{L}(k,a)=L)[P(\widetilde{a}_{k,L}=1)|\widehat{\Phi}_{b_{k,L,1}}(\theta)| + P(\widetilde{a}_{k,L}=-1)|\widehat{\Phi}_{b_{k,L,-1}}(\theta)|] \\ &\le 1 - \frac{\theta^2}{\pi^2}\min_{1\le k\le d}\sum_{L=1}^{d} P(\mathfrak{L}(k,a)=L)[P(\widetilde{a}_{k,L}=1)\mathtt{Var}(b^{(n)}_{k,\ell,1}) + P(\widetilde{a}_{k,L}=-1)\mathtt{Var}(b^{(n)}_{k,\ell,-1})] \\ &= 1 - \frac{\kappa_{F_n}\theta^2}{\pi^2} \\ &\le 1 - \beta\theta^2 \end{aligned}$$

for any $0 < \beta \le \min_{n\ge1}\frac{\kappa_{F_n}}{\pi^2}$ which latter is positive by uniformity. ☑(i)

Proof of (ii) and (iii) Statement (ii) follows from adaptedness. Statement (iii) can be obtained by shrinking β. ☑(ii), (iii) and ¶.

To complete the proof of (b), we have

$$\begin{aligned} |\widehat{\Phi}_{X^{(n)}_k}(\theta)| &= |(P_{F_n}(\theta)P_{F_{n-1}}(\theta)\cdots P_{F_1}(\theta)1)_k| \\ &\le \prod_{k=1}^{n}\|P_k(\theta)\|. \end{aligned}$$

By ¶(i), for $n \ge 1,\ \gamma \in \Gamma_{F_n}$ we have

$$\|P_{F_n}(\gamma+\theta)\| \le 1 - \beta\theta^2\quad \forall\ |\theta| < \Delta$$

and by ¶(ii)

$$\|P_{F_n}(t)\| \le \rho\quad \forall\ t \notin B(\Gamma_{F_n},\Delta).$$

It follows that for $\gamma \in \Gamma \setminus B(\Gamma_{F_n}, \Delta)$ and $|\theta| < \Delta$, we have

$$\begin{aligned}\|P_{F_n}(\gamma+\theta)\| &\le \rho \quad \text{by ¶(ii)}\\ &\le 1-\beta\theta^2 \quad \text{by ¶(iii)}.\end{aligned}$$

Thus for $\gamma \in \Gamma$:

$$|\widehat{\Phi}_{X_k^{(n)}}(\gamma+\theta)| \le (1-\beta\theta^2)^n \qquad |\theta| < \Delta$$

and

$$|\widehat{\Phi}_{X_k^{(n)}}(t)| \le \rho^n \quad \forall\ t \notin B(\Gamma, \Delta),$$

whence

$$\begin{aligned}\int_{\mathbb{T}} |\widehat{\Phi}_{X_k^{(n)}}(\theta)| d\theta &\le \Big(\int_{B(\Gamma,\Delta)} + \int_{\mathbb{T}\setminus B(\Gamma,\Delta)}\Big) \prod_{k=1}^n \|P_k(\theta)\| d\theta\\ &\le \sum_{\gamma\in\Gamma} \int_{-\Delta}^{\Delta} \prod_{k=1}^n \|P_k(\gamma+\theta)\| d\theta + \int_{\mathbb{T}\setminus B(\Gamma,\Delta)} \prod_{k=1}^n \|P_k(\theta)\| d\theta\\ &\le N \int_{-\Delta}^{\Delta} (1-\beta\theta^2)^n d\theta + \rho^n\\ &\le \frac{N}{\sqrt{n}} \int_{-\Delta\sqrt{n}}^{\Delta\sqrt{n}} \Big(1-\frac{\beta t^2}{n}\Big)^n dt + \rho^n\\ &\le \frac{N}{\sqrt{n}} \int_{\mathbb{R}} e^{-\beta t^2} dt + \rho^n\\ &\propto \frac{1}{\sqrt{n}}. \quad \text{☑(b) and } \texttt{WRLLT}\end{aligned}$$

§7 Centering of a RAT sequence

Let $(F_n = (a^{(n)}, b^{(n)}) :\ n \ge 1)$ be an independent RAT sequence and let $X^{(n)} := F_n \circ F_{n-1} \circ \cdots \circ F_1(0)$ and $c_n := E(X^{(n)})$. Set $\widetilde{X}^{(n)} = X^{(n)} - c_n$, then

$$\begin{aligned}\widetilde{X}^{(n)} &= X^{(n)} - c_n\\ &= a^{(n)} X^{(n-1)} + b^{(n)} - c_n\\ &= a^{(n)} \widetilde{X}^{(n-1)} + b^{(n)} - c_n + a^{(n)} c_{n-1}\\ &=: \widetilde{F_n}(\widetilde{X}^{(n-1)})\\ &= \widetilde{F}_n \circ \widetilde{F}_{n-1} \circ \cdots \circ \widetilde{F}_1(0).\end{aligned}$$

and so $(\widetilde{X}^{(n)})_{n\ge1}$ is a centered ARW with the corresponding centered independent, RAT sequence $(\widetilde{F}_n = (a^{(n)}, \widetilde{b}^{(n)}) :\ n \ge 1)$ where

$$\widetilde{b}^{(n)} = b^{(n)} - c_n + a^{(n)} c_{n-1} = 0.$$

The ARW and associated RAT sequence above are unique. We call them the *centerings* of the ARW and associated RAT sequence respectively.

Proposition 7.1 *Let $(X^{(n)} :\ n \ge 1)$ be an* ARW *with centering $(\widetilde{X}^{(n)})_{n\ge1}$, then for each $n \ge 1$ and $1 \le k \le d$,*

$$\mathtt{Var}(X_k^{(n)}) = E(\widetilde{X}_k^{(n)2}).$$

Adaptedness preservation.

The centering of a discrete ARW may not be discrete, but centering does not affect adaptedness or any of the other compactness properties.

This is because if $(F_n :\ n \ge 1)$ is a RAT with centering $(\widetilde{F}_n :\ n \ge 1)$, then

$$\widetilde{F}_n = (a(F_n), b(F) - c^{(n)} + a(F) c^{(n-1)})$$

where $c^{(n)} \in \mathbb{R}^d$ $n \ge 1$ are constant.

Thus, for $1 \le k, \ell \le d$ and $\epsilon = \pm 1$ with $P(a_{k,\ell} = \epsilon) > 0$, the random variable $\widetilde{b}_{k,\ell,\epsilon}$ is a constant translation of $b_{k,\ell,\epsilon}$:

$$\widetilde{b}_{k,\ell,\epsilon}(\widetilde{F}_n) = b_{k,\ell,\epsilon}(F_n) - c_k^{(n)} + \epsilon c_\ell^{(n-1)}.$$

It follows from ☼that the adaptedness of $(F_n :\ n \ge 1)$ is equivalent to that of $(\widetilde{F}_n :\ n \ge 1)$ and the adaptivity groups are the same.

We'll need conditions for $\sup_{n\ge1} \|b^{(n)} - \widetilde{b}^{(n)}\| < \infty$. a.s..

Bounded mean fluctuation.

We say that the ARW $(X^{(n)} := F_n \circ F_{n-1} \circ \cdots \circ F_1(0) :\ n \ge 1)$ (and its associated RAT sequence) has *bounded mean fluctuations* BMF if

$$\sup_{n\ge1} \|E(X^{(n)})\| < \infty.$$

7.2 Bounded Centering Proposition *Let $(F_n = (a^{(n)}, b^{(n)}) :\ n \ge 1)$ be a* RAT *sequence with centering $(\widetilde{F_n} = (a^{(n)}, \widetilde{b^{(n)}}) :\ n \ge 1)$.*

If $(F_n = (a^{(n)}, b^{(n)}) :\ n \ge 1)$ has BMF*, then $\exists\ M > 1$ so that*

$$\|\widetilde{b^{(n)}} - b^{(n)}\| \le M \quad a.s. \quad \forall\ n \ge 1.$$

Proof Since

$$\widetilde{b}^{(n)} - b^{(n)} - c_n + a^{(n)} c_{n-1}$$

and $\|a^{(n)}\| \le 1$ a.s., the proposition holds with

$$M = 2\sup_{n\ge 1}\|E(X^{(n)}\|. \quad \boxtimes$$

The following gives sufficient conditions for `BMF` of a `RAT` sequence in terms of uniform mean boundedness and uniform mean contractivity.

7.3 Lemma *Let $(F_n = (a^{(n)}, b^{(n)}):\ n \ge 1)$ be a* `RAT` *sequence. If $\exists\ M > 0$ and $\rho \in (0,1)$ so that*

$$\|E(b^{(k)})\| \le M \text{ and } \|E(a^{(k)})\| \le \rho\ \forall\ k \ge 1,$$

then $(F_n = (a^{(n)}, b^{(n)}):\ n \ge 1)$ has `BMF`.

Proof

$$X^{(n)} := F_n \circ F_{n-1} \cdots \circ F_1(0) = \sum_{\nu=1}^{n} a_{\nu+1}^{n} b^{(\nu)}$$

where

$$a_k^N := \begin{cases} a^{(N)}a^{(N-1)}\cdots a^{(k)} & k \le N \\ \mathrm{Id} & k > N, \end{cases}$$

we have by independence that

$$E(X^{(n)}) = \sum_{\nu=1}^{n} E(a_{\nu+1}^{n})E(b^{(\nu)}) = \sum_{\nu=1}^{n}\prod_{k=\nu+1}^{n} E(a^{(k)})E(b^{(\nu)})$$

whence

$$\|E(X^{(n)})\| \le \sum_{\nu=1}^{n}\prod_{k=\nu+1}^{n}\|E(a^{(k)})\|\|E(b^{(\nu)})\| \le M\sum_{k=1}^{\infty}\rho^k = \frac{M\rho}{1-\rho} < \infty. \quad \boxtimes$$

§8 Proof of the main result

Our first step in the proof of (🙰) (as on page 2) is to show that for $\alpha \in$ `BAD`, each α-`ARW` satisfies the `WRLLT` along a syndetic sybsequence.

A canonical subsequence. Define $\nu : \mathbb{N}_2^{\mathbb{N}} \to \mathbb{N} \cup \{\infty\}$ by

$$\nu(n_1, n_2, \dots) := \min\{J \ge 1 :\ \sum_{k=1}^{J} 1_{[n_k > 2]} \ge 4\}.$$

The number $\beta = 2\alpha = [n_1, n_2, \dots]$ is irrational iff $\#\{k \ge 1 :\ n_k > 2\} = \infty$ so $\nu(n_{k+1}, n_{k+2}, \dots) < \infty\ \ \forall\ k \ge 1$.

Define

$$\nu_0 = 0,\ \ \nu_{k+1} := \nu_k + \nu(n_{\nu_k+1}, n_{\nu_k+2}\dots),$$

then $\nu_k < \infty\ \forall\ k \ge 1$.

By [15], $\alpha \in \mathtt{BAD}$ iff

(i) $\sup_{k\ge 1} n_k < \infty$ and (ii) $\sup\{J \ge 1 :\ \exists\ k,\ n_{k+j} = 2\ \forall\ 1 \le j \le J\} < \infty$.

Thus in case $\alpha \in \mathtt{BAD}$, $\sup_{k\ge 1}(\nu_{k+1} - \nu_k) < \infty$ i.e. $\{\nu_k\}_{k\ge 1}$ is syndetic..

The grouping theorem. Let $(F_n = (a^{(n)}, b^{(n)}) :\ n \ge 1)$ be a α-$\mathtt{RAT}$ sequence. The *canonical grouping* of α (or $(F_n :\ n \ge 1)$) is the $\mathtt{RAT}$ sequence $(G_n :\ n \ge 1)$ defined by

$$G_0 = \mathtt{Id},\ G_{k+1} := F_{\nu_{k+1}} \circ F_{\nu_{k+1}-1} \circ \cdots \circ F_{\nu_k+1}.$$

8.1 Grouping theorem *Let $\alpha \in \mathtt{BAD}$, let $(F_k :\ k \ge 1)$ be the associated $\mathtt{RAT}$, then the canonical grouping $(G_k :\ k \ge 1)$ of $(F_k :\ k \ge 1)$ has $\mathtt{BMF}$ and is adapted.*

The proof of the Grouping theorem proceeds via:

Compact $\mathtt{RAT}$ collections.

We'll call the collection $\mathfrak{F} \subset \mathtt{RV}(M_{d\times d}(\mathbb{Z}^d) \times \mathbb{Z}^d)$ of flip type $\mathtt{RAT}$s *compact* if

$\exists\ M = M_{\mathfrak{F}} > 1$ so that $\forall\ F := (a,b) \in \mathfrak{F}$ and
$1 \le k, \ell \le d,\ \epsilon = 0, \pm 1,\ \nu \in \mathbb{Z}$ we have

(i) $P([a_{k,\ell} = \epsilon] \cap [b_k = \nu]) > 0 \implies P([a_{k,\ell} = \epsilon] \cap [b_k = \nu]) \ge \frac{1}{M}$ and

(ii) $\|b\|_\infty := \sup\limits_{P(b=\beta)>0} \|\beta\| \ \le\ M.$

We'll call the $\mathtt{RAT}$ sequence $(F_k :\ k \ge 1)$ *compactly generated* if the collection $\{F_k :\ k \ge 1\}$ is compact.

8.2 Compactness lemma *Suppose that $2\alpha = [n_1, n_2, \ldots] \in \mathtt{BAD}$ with*

$$\sup_{k\ge 1} n_k =: M \text{ and } \max\{J \ge 1 :\ \exists\ k,\ n_{k+i} = 2\ \forall\ 1 \le i \le J\} =: \mathfrak{r}$$

and let $(F_n :\ n \ge 1\}$ be an α-$\mathtt{RAT}$, then

$$\mathfrak{F} := \{F_{j_1} \circ F_{j_2} \circ \cdots \circ F_{j_r} :\ j_1, j_2, \ldots, j_r \in \mathbb{N} \text{ and } 1 \le r \le \mathfrak{r}\}$$

is compact.

Before proving the compactness lemma, we need a

8.3 Sublemma *If $\alpha \in \mathtt{BAD}$, then $\exists\ \Delta > 0$ so that*

(i) $r_k := \frac{\ell_k(1)}{\ell_k(0)} \ge \Delta\ \forall\ k \ge 1,$ *and*

(ii) $\frac{p_k(i)}{1-p_k(i)} \in [n_{k+1} - 1 - i, \frac{n_{k+1}-1-i}{\Delta}]$.

Proof of (i)

$$r_{k+1} = \frac{\ell_{k+1}(1)}{\ell_{k+1}(0)} = \frac{(n_{k+1}-2)\ell_k(0)+\ell_k(1)}{(n_{k+1}-1)\ell_k(0)+\ell_k(1)} = 1 - \frac{1}{n_{k+1}-1+r_k}.$$

In case $n_{k+1} \ge 3$, we have

$$r_{k+1} \ge \frac{1}{2}.$$

When $n_{k+1} = 2$, $r_{k+1} = v(r_k)$ where $v(x) := \frac{x}{1+x}$. Now suppose that $n_k \ge 3$ and $n_{k+1} = n_{k+2} = \cdots = n_{k+R} - 2$, then for $\forall\ 1 \le j \le R$,

$$r_{k+j} = v(r_{k+j-1}) = v^j(r_k) \ge v^j(\tfrac{1}{2}).$$

Inspection of this recursion shows that

$$\frac{1}{r_{k+j}} = \frac{1}{r_k} + j \le 2 + j \quad \forall\ 1 \le j \le R.$$

Let $\mathfrak{r} \in \mathbb{N}$ be so that $\forall\ k\ \exists\ J \le \mathfrak{r}+1,\ n_{k+J} \ge 3$ and set $\Delta := \frac{1}{2+\mathfrak{r}}$. It follows that

$$r_k \ge \Delta. \quad \boxtimes$$

proof of (ii)

$$\frac{p_k(i)}{1-p_k(i)} = \frac{(n_{k+1}-1-i)\ell_k(0)}{\ell_k(1)} = \frac{n_{k+1}-1-i}{r_k} \in [n_{k+1}-1-i, \frac{n_{k+1}-1-i}{\Delta}]. \quad \boxtimes$$

Proof of the compactness lemma

We claim first that the RAT collection $\mathcal{F}_0 := \{F_n :\ n \ge 1\}$ is compact.

Condition (ii) is immediate from the boundedness of $\{n_k :\ k \ge 1\}$. To see (i), note first that

- $q_2 = 1$ and $q_N \in [\frac{1}{3}, \frac{1}{2}]\ \forall\ N \ge 3$,
- $p_k(1) = 0$ when $n_k = 2$ and by the sublemma
- $\exists\ R > 0$ so that $p_k(i),\ 1-p_k(i) \ge R\ \forall\ k \ge 1,\ i = 0,1,\ (n_k, i) \ne (2,1)$.

Next, for each fixed $k \ge 1,\ i,j = 0,1$ and $\nu \in \mathbb{Z}$,

$$P([a_{i,j}(F_k) = \epsilon] \cap [b_i(F_k) = \nu])$$

is a polynomial of degree at most 3 in

$$\underline{z}^{(k)} = (z_1^{(k)}, \dots, z_9^{(k)}) = (p_k(0),\ 1-p_k(0),\ q_{n_k},\ 1-q_{n_k}, p_k(1),\ 1-p_k(1),\ q_{n_k-1},\ 1-q_{n_k-1}, \frac{1}{n_k})$$

whose coefficients are non-negative integers. By sublemma 8.3, $\exists\ \epsilon > 0$ so that

$$1 \le s \le 9,\ k \ge 1,\ z_s^{(k)} > 0 \implies z_s^{(k)} \ge \epsilon.$$

Condition (i) follows because if

$$P([a_{i,j}(F_k) = \epsilon] \cap [b_i(F_k) = \nu]) = F(\underline{z}) = \sum_{1 \le r,s,t \le 9} N_{r,s,t} z_r z_s z_t > 0$$

then $\exists\ 1 \le r, s, t \le 9$ (maybe not distinct) so that $N_{r,s,t} \ge 1$ and $z_r,\ z_s,\ z_t \ge \epsilon$. This shows that $\mathcal{F}_0$ is compact.

For analogous reasons, the compactness persists among concatenations of $\{F_n :\ n \ge 1\}$ of length bounded by $\mathfrak{r}$.

Let $F = F_{k_r} \circ F_{k_{r-1}} \circ \dots \circ F_{k_1} \in \mathfrak{F}$, then

$$a(F) = a(F_{k_r}) a(F_{k_{r-1}}) \dots a(F_{k_1}) \text{ and } b(F) = \sum_{\nu=1}^{r-1} a(F_{\nu+1}^r) b(F_{k_\nu})$$

where $F_{\nu+1}^r := F_{k_r} \circ F_{k_{r-1}} \circ \dots \circ F_{k_{\nu+1}}$.

Condition (ii) is immediate since

$$\|b(F)\|_\infty \le \sum_{\nu=1}^{r-1} \|a(F_{\nu+1}^r) b(F_{k_\nu})\|_\infty \le \sum_{\nu=1}^{r-1} \|b(F_{k_\nu})\|_\infty \le \mathfrak{r} \sup_n \|b(F_n)\|_\infty.$$

To see (i) we note that for fixed $i, j = 0, 1$ and $\nu \in \mathbb{Z}$,

$$P([a_{i,j}(F) = \epsilon] \cap [b_i(F) = \nu])$$

is a polynomial of degree at most most r in the variables

$$\{P([a_{k,\ell}(F_{k_u}) = \omega] \cap [b_k(F_{k_u}) = \mu]) :\ u = 1, 2, \dots, r,\ \omega = 0, \pm 1,\ \mu \in \mathbb{Z}\}$$

whose coefficients are non-negative integers and so

$$P([a_{i,j}(F) = \epsilon] \cap [b_i(F) = \nu]) > 0 \implies$$

$$P([a_{i,j}(F) = \epsilon] \cap [b_i(F) = \nu]) \ge \frac{1}{M_{\mathfrak{F}_0}^{\mathfrak{r}}}$$

where $M_{\mathfrak{F}_0}$ is as in the definition of compactness of $\mathcal{F}_0$. This proves the compactness lemma. ☑

Proof of the Grouping theorem.

In order to prove that $(G_n : n \geq 1)$ has `BMF`, it suffices by lemma 7.3 and the compactness lemma, to show that each G_n is mean contractive.

We claim also that in order to prove adaptedness of the sequence $(G_n : n \geq 1)$, it suffices to show that each `RAT` G_n is adapted.

To prove this latter claim, suppose that each G_n is individually adapted. Compactness of $(G_n : n \geq 1)$ entails $\|b(G_n)\| \leq M$ which implies by discreteness that (possibly increasing M) $\forall\ k \geq 1$, `supp` $b_k(G_n) \subset [-M, M]$ whence

$$\Gamma_{G_n} \subset \frac{2\pi}{(2M)!}\mathbb{Z}.$$

Adaptedness of the sequence now indeed follows individual adaptedness by compactness.

The rest of this proof is concerned with establishing these individual properties for the concatenations involved.

Concatenations of `spec-RAT`s. Each $G = G_n$ in the canonical grouping is an independent concatenation of four `RAT`s of form

$$H = R \circ T$$

where R, T are independent, $R = R(H)$ is a non-trivial `spec-RAT` and either $T = T(H)$ is a concatenation of finitely many independent, trivial `spec-RAT`s, or $T =$ `Id`.

Mean contractivity of `spec-RAT` concatenations

Non-trivial, odd `spec-RAT`s are mean contractive. Even *spec-RAT*s are not. Suppose $G = H_1 \circ H_2$ where and $R(H_i)$ $i = 1, 2$ are even. By the parity transition laws, each $R(H_i)$ is followed and preceded by odd (here necessarily trivial) `spec-RAT`s J. It can be calculated that for $a = a(J \circ R)$ or $a = a(R \circ J)$, then $P(a_{i,0} = \epsilon) > 0$ for $i = 0, 1$ and $\epsilon = \pm 1$. By irreducibility G is mean contractive.

Adaptedness of `spec-RAT` concatenations

If $G = H_1 \circ H_2 \circ H_3 \circ H_4$ where $R(H_i)$ is even for some $i = 2, 3, 4$, then $R(H_i)$ is partially adapted and $H_1 \circ \cdots \circ H_{i-1}$ is irreducible, whence H_1, $H_1 \circ \cdots \circ H_{i-1} \circ R(H_2)$ is adapted and so is G.

The remaining case is where $G = H_1 \circ \cdots \circ H_4$ where $R(H_i)$ is odd $\forall\ 2 \leq i \leq 4$.

To show that such G is adapted, we write

$$G = H_1 \circ U \circ T$$

where

$$U := H_2 \circ H_3 \circ R(H_4).$$

Since H_1 is irreducible, it suffices to show that row 0 is adapted for either $H_2 \circ H_3$ or for U.

Suppose that row 0 is not adapted for $H_2 \circ H_3$, then, in particular the random variable $b_{0,0,1}(H_2 \circ H_3) \equiv c$ is constant.

For each $\alpha = \pm 1$, $P(J_\alpha) > 0$ where

$$J_\alpha = [a_{0,0}(H_2) = 1] \cap [a_{0,0}(H_3) = \alpha] \cap [a_{0,0}(R(H_4)) = \alpha] \subset [a_{0,0}(U) = 1]$$

and, on J_α we have that

$$b_0(U) = b_0(H_2 \circ H_3) + \alpha b_0(R(H_4)) = c_1 + \alpha.$$

Thus the random variable $P(b_{0,0,1}(U) = c_1 + \alpha) > 0$ for $\alpha = \pm 1$ and row 0 is adapted for U. ☑

This completes the proof of the Grouping theorem. □

Conclusion of the proof of the main result

By the adaptedness preservation observation, the centering $(H_k : k \geq 1)$ of $(G_k : k \geq 1)$ is adapted (with the same adaptivity group).

By compactness, `BMF` and the bounded centering lemma, $(H_k : k \geq 1)$ is uniform.

Thus $(H_k : k \geq 1)$ satisfies `WRLLT`. Let $(X_n : n \geq 1)$, $(Y_k : k \geq 1)$ and $(Z_k : k \geq 1)$ be the `ARW`s generated by $(F_n : n \geq 1)$, $(G_k : k \geq 1)$ and $(H_k : k \geq 1)$ respectively, then for each $i = 0, 1$ and $1 \leq p \leq 2$,

$$\begin{aligned}\int_{\mathbb{T}} |\widehat{\Phi}_{X^{(\nu_k)}(i)}(\theta)|^p d\theta &= \int_{\mathbb{T}} |\widehat{\Phi}_{Y^{(k)}(i)}(\theta)|^p d\theta \quad \text{because the processes are identical} \\ &= \int_{\mathbb{T}} |\widehat{\Phi}_{Z^{(k)}(i)}(\theta)|^p d\theta \ \ \text{because } Z^{(k)}(i) = Y^{(k)}(i) - E(Y^{(k)}(i)) \\ &\asymp \frac{1}{\sqrt{k}} \ \ \text{by the } \mathtt{WRLLT}.\end{aligned}$$

By the Visit Lemmas 2.4,

$$\int_{\mathbb{T}} \Psi_{\ell_{\nu_k}(0)}(x)dx \ \gg \ \ell_{\nu_k}(1) \int_{\mathbb{T}} |\widehat{\Phi}_{X^{(\nu_k)}(1)}(\theta)|^2 d\theta \asymp \frac{\ell_{\nu_k}(1)}{\sqrt{k}},$$

and

$$\|\Psi_{\ell_{\nu_k}(1)}\|_\infty \leq \ell_{\nu_k(0)} \int_{\mathbb{T}} |\widehat{\Phi}_{(X^{(\nu_k)}(0)}(\theta)| d\theta \asymp \frac{\ell_{\nu_k}(0)}{\sqrt{k}}.$$

Next, since $\alpha \in \mathtt{BAD}$ and $\sup_{k\ge 1}(\nu_{k+1} - \nu_k) < \infty$, $\exists\ 1 < m < M$ so that

$$m \le \frac{\ell_{\nu_{k+1}}(i)}{\ell_{\nu_k}(j)} \le M \quad i, j = 0, 1$$

whence

$$\log \ell_{\nu_k}(j) \asymp k.$$

For $\ell_{\nu_k}(0) \le n \le \ell_{\nu_{k+1}}(0)$,

$$\int_{\mathbb{T}} \Psi_n(x)dx \ge \int_{\mathbb{T}} \Psi_{\ell_{\nu_k}(0)}(x)dx \gg \frac{\ell_{\nu_k}(1)}{\sqrt{k}} \gg \frac{\ell_{\nu_k+1}(1)}{\sqrt{k}} \gg \frac{n}{\sqrt{\log n}}$$

and for $\ell_{\nu_k}(1) \le n \le \ell_{\nu_k+1}(1)$,

$$\|\Psi_n\|_\infty \ll \frac{\ell_{\nu_{k+1}}(1)}{\sqrt{k}} \ll \frac{\ell_{\nu_k}(0)}{\sqrt{k}} \ll \frac{n}{\sqrt{\log n}}. \quad ☑(🚂) \text{ (as on page 2)}$$

Concluding Remarks.

1. It is shown in [7] that for a.e. $\alpha \in \mathbb{T}$, T_α is $\frac{1}{n}$-recurrent in the sense that $\sum_{n=1}^\infty \frac{1}{n} f \circ T_\alpha^n = \infty$ a.e. $\forall\ f \in L^1_+$.

It follows easily from (🚂) (as on page 2) that for $\alpha \in \mathtt{BAD}$ and $\omega_n \downarrow 0$: T_α is ω_n-recurrent ($\sum_{n=1}^\infty \omega_n f \circ T_\alpha^n = \infty$ a.e. $\forall\ f \in L^1_+$) iff

$$\sum_{n\ge 1} \frac{n(\omega_n - \omega_{n+1})}{\sqrt{\log n}} = \infty$$

e.g. $\omega_n = \frac{1}{n\sqrt{\log n}}$.

2. The subsequence condition (⊕) (as on page 2) is satisfied for some $\alpha \notin \mathtt{BAD}$. For example, if $2\alpha = [2m_1 + 1, 2m_2, 2m_3, \dots]$ with $m_k \ge 2\ \forall\ k \ge 1$, then by sublemma 8.3(i), $\ell_k(1) \le \ell_k(0) \le 2\ell_k(1)$ and it follows as above that (⊕) is satisfied along $\ell_{\mu_k}(0)$ ($\mu_k \uparrow \infty$) as soon as

$$\int_{\mathbb{T}} |\widehat{\Phi}_{X^{(\mu_k)}(0)}(\theta)| d\theta \ll \int_{\mathbb{T}} |\widehat{\Phi}_{X^{(\mu_k)}(1)}(\theta)|^2 d\theta.$$

The parities $\epsilon_k(0) = 1\ \forall\ k \ge 2$, whence the associated $\mathtt{RAT}$ collection $\{F_j :\ j \ge 1\}$ is compact.

The canonical subsequence in §8 is $\nu_k = 1 + 4k \quad (k \ge 1)$, the grouping lemma applies and the centering of the canonical grouping $\mathtt{RAT}$ sequence satisfies the $\mathtt{WRLLT}$.

Consequently T_α satisfies (⊕) along $\ell_{\nu_k}(0)$ with $a_{\ell_{\nu_k}} \asymp \frac{\ell_{\nu_k}}{\sqrt{k}}$.

3. It is thus natural to ask if $p(⊕) = 1$ (or even if $p(⊕) > 0$) where

$$p(⊕) := m(\{\alpha \in (0,1) :\ T_\alpha \text{ satisfies (⊕) along some subsequence}\}).$$

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(Aaronson) School of Math. Sciences, Tel Aviv University, 69978 Tel Aviv, Israel.

Webpage : `http://www.math.tau.ac.il/~aaro`

E-mail address: `aaro@post.tau.ac.il`

(Bromberg) School of Math. Sciences, Tel Aviv University,r 69978 Tel Aviv, Israel.
E-mail address: mic1@post.tau.ac.il

(Nakada) Dept. Math., Keio University,Hiyoshi 3-14-1 Kohoku, Yokohama 223, Japan
E-mail address, Nakada: nakada@math.keio.ac.jp